Dmitry Voloshyn

Geometric quantization

Dirac's axioms

countable basis

$\mathcal{H}$ separable complex Hilbert space Ex: $L^2([0,1],\mathbb{C})$

$S(\mathcal{H})$ - densely defined symmetric endomorphisms of $\mathcal{H}$

($A \in S(\mathcal{H}) \iff$ domain(A) is dense in $\mathcal{H}$, $(x, Ay) = (Ax, y)$)

it is more general than self-adjoint.

Ex: $A =$ second derivative
- unbounded operator
- symmetric but not self-adjoint

(M, ω) - symp. mfd.

def $\{f_1 \dots f_n\} \subset C^\infty(M)$ is complete

if $g \in C^\infty(M)$ $\{f_i, g\}_M = 0$ $\forall i$, then $g = $ const

def: $\{A_1, \dots, A_n\} \subset S(\mathcal{H})$ is complete

if $B \in S(\mathcal{H})$ s.t. $[B, A_i] = 0$ $\forall i$, then $B = c \cdot \mathbb{1}$.

Dirac's Axioms:

def: A quantization is a linear map $Q: C^\infty(M) \longrightarrow S(\mathcal{H})$ s.t.

1) $Q(1) = \mathbb{1}$
2) $-i\hbar\, Q(\{f,g\}) = [Q(f), Q(g)]$
3) Whenever $\{f_1, \dots, f_n\}$ is complete
$\{Q(f_1), \dots, Q(f_n)\}$ is complete

Thm (Groenewold, Van Hove)

$\nexists\, Q$ - quantization does not exist.

Talk 2

$A : \mathcal{D}(A) \to H$ is symm if $\overline{\mathcal{D}(A)} = H$
$\cap$
H

and $(x, Ay) = (Ax, y)$
$\forall x, y \in \mathcal{D}(A)$

$A : D(A) \subseteq H \to H$ is self-adjoint
$A = A^*$

$x \in \mathcal{D}(A^*) \iff \exists z \in H \quad \forall y \in \mathcal{D}(A)$
$(Ay, x) = (y, z)$

if A is symm. then $A \subset A^*$ $\{\mathrm{Graph}_A \subset \mathrm{Graph}_{A^*}\}$

Ex: $L^2([0,1], \mathbb{R})$, $A = -\frac{d^2}{dx^2}$ sym. but not self-adjoint

(M, ω) symp. mfd $\quad$ H separable $\overset{\mathbb{C}}{\text{Hilbert}}$ space
$\uparrow$
countable basis (closure of the span of basis is H)

$S(H)$ densely defined symmetric operators.

<u>def</u> A quantization map is a linear map $C^\infty(M) \to S(H)$ s.t.

(1) $Q(1) = \mathrm{id}$

(2) $-i\hbar\, Q(\{f,g\}_M) = [Q(f), Q(g)]$

(3) $\{f_1, \dots, f_n\}$ – complete set of functions on M
$\Rightarrow$ $\{Qf_1, \dots, Qf_n\}$ <u>complete</u>.

<u>Thm</u> (Groenewold, Van Hove): A quantization <u>does not exist</u>. (– problem in condition (3))

some small Lie subalgebras can be quantized

If Q satisfies only (1), (2), it is called a "pre-quantization"

<u>Segal's prequantization</u>

(T^*M, ω) $\quad$ $\omega = d\Theta$
$\quad$ $\leftarrow$ std symp potential (taut 1-form)

$H = L^2(T^*M, \mathbb{C})$

$$\boxed{Q(f) = -i\hbar X_f + f - \iota_{X_f}\Theta} =: \hat{f}$$

If we change $\Theta' = \Theta + d\varphi$

then $Q(f)$ change by $-\iota_{X_f} d\varphi$

A way to circumvent this issue.

If we change the potential $\to$ then we change $\underset{\hat{\mathcal{H}}}{\psi} \mapsto \psi' = e^{\frac{i\varphi}{\hbar}}\psi$.

$\hat{f} = -i\hbar X_f + f - \iota_{X_f}\Theta$ — with old symp potential

$\hat{f}' = -i\hbar X_f + f' - \iota_{X_f}\Theta'$ — with new symp potential

then: $e^{i\varphi/\hbar}\hat{f}(\psi) = \hat{f}'(\psi')$ — equivariance

For a general (M, ω), cannot find a global symp. potential

$\to$ promote it to a connection in a "pre-quantum line bundle".

Kostant - Souriau prequantization

def: a prequantum line bundle ($\mathcal{L}$ is a Hermitian line bundle $E \to M$ on (M,ω)) with a compatible connection ∇ with curvature $\hbar^{-1}\omega$.

def $Q: C^\infty(M) \to S(\Gamma_{L^2}(M, \mathcal{L}))$

$$Q(f) = -i\hbar \nabla_{X_f} + f \qquad \langle s, s' \rangle = \int_M \langle s, s' \rangle \frac{\omega^n}{n!}$$

Def Weil integrality condition

(1) $\frac{1}{2\pi\hbar}\int_\Sigma \omega \in \mathbb{Z}$

$\Sigma \leftarrow$ any closed surface

(2) $[\omega] \in H^2(M, \mathbb{R}) \Rightarrow [\omega] \in H^2_{2\pi\hbar \text{int}}(M, \mathbb{R})$

$\uparrow$ image of $H^2(M, \mathbb{Z})$

Thm a prequantum line bundle exists iff ω is $\hbar$-integral.

How to construct this line bundle from an $\hbar$-integral ω?

Recall: $H^2(M,\mathbb{R}) \cong \check{H}^2(\mathcal{U},\mathbb{R})$ Čech cohom.

$\mathcal{U} = \{U_i\}$ open cover by contractible opens

$[\omega] \longmapsto [f]$

On each U_i, choose a symp potential $\omega = d\theta_i$

$\theta_i - \theta_j = du_{ij}$ on $U_i \cap U_j \neq \emptyset$

$f_{ijk} := u_{ij} + u_{jk} + u_{ki}$ on $U_i \cap U_j \cap U_k \neq \emptyset$

– locally constant functions

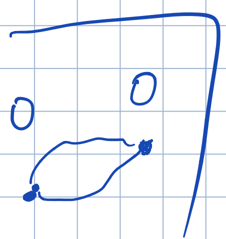
f_{ijk} satisfy Čech cocycle condition

$f_{jik} - f_{jkl} + f_{kli} - f_{lij} = 0$

ω $\hbar$-integral $\Rightarrow \frac{1}{2\pi\hbar} f_{ijk} \in \mathbb{Z}$

transition functions $c_{ij} = e^{i u_{ij}/\hbar}$

$c_{ij} c_{jk} c_{ki} = e^{\frac{i}{\hbar} f_{ijk}} = 1.$



2nd talk

$$H^2_{dR}(M,\mathbb{R}) \cong \check{H}(\mathcal{U}, \mathbb{R})$$

good cover

$$[\omega] \longrightarrow [f]$$

on each U_i $\quad \omega = d\theta_i$

on each $U_i \cap U_j \neq \emptyset \quad \theta_i - \theta_j = dU_{ij}$

$f_{ijk} = U_{ij} + U_{jk} + U_{ki} \quad$ on $U_i \cap U_j \cap U_k \neq \emptyset$

ω is $\hbar$-integral $\quad$ iff $\frac{1}{2\pi\hbar} f_{ijk} \in \mathbb{Z}$

Thm A prequantum line bundle H exists iff ω is $\hbar$-integral

Proof (sketch)

The transition functions are given by $c_{ij} = \exp \frac{i u_{ij}}{\hbar}$

then $c_{ij} c_{jk} c_{ki} = \exp \frac{i f_{ijk}}{\hbar} = 1$

On each U_i, choose s_i

$$s_i(x) = (x, 1) \in U_i \times \mathbb{C}$$

related via $\quad c_{ij} s_j = s_i$

$$\frac{dc_{ij}}{c_{ij}} = \frac{i}{\hbar} du_{ij} = \frac{i}{\hbar}(\theta_i - \theta_j) \qquad (*)$$

$$\nabla^i_X(s_i) = \frac{i}{\hbar}\, \theta_i(X)\, s_i$$

$(*)$ is precisely the compatibility condition for ∇^i's

Ex: (T^*M, ω)

for any closed 2-surface Σ, $\frac{1}{2\pi\hbar}\int_\Sigma \omega = \frac{1}{2\pi\hbar}\int_{\partial\Sigma} \Theta = 0$

prequantum space of states $L^2_{\mathbb{C}}(T^*M)$ $\quad$ - prequantum operators act on it

want to "cut it in half"

Polarizations

def A real polarization is an involutive distribution $P \subset TM$ s.t. $P_m \subset T_m M$ is Lagrangian.

Ex: $\mathbb{R}^{2n}$ $\omega = dq^i \wedge dp_i$ polarizations: ① q^i fixed
② p_i fixed

Ex: (T^*M, ω)
Foliation is given by the fibers of T^*M

Ex: $S^1 \times S^1$
Foliations: $S^1 \times \{\theta\}$
$\{\varphi\} \times S^1$

Ex: S^2 (with ω = area form)
There is no polarization!
If P is a real line bundle $/S^2$ $\Rightarrow$
$\Rightarrow$ P is trivial $\Rightarrow$ $\exists$ non vanishing section not allowed to exist on S^2
(motivates cx polarizations)

Prop'n: Any real pol. is locally iso to stand. polarization on $\mathbb{R}^{2n}$.

Complex & Kähler polarizations

def: (M, ω) $P \subseteq T_{\mathbb{C}}M$ is a cx pol if
1) $P_m \subset T_{\mathbb{C}} M|_m$ is Lagn
2) $\dim(P \cap \bar{P} \cap TM) = \text{const}$
3) P is integrable
($\exists f_1 \dots f_n$ cx valued fun. s.t. their $\mathrm{Span}(X_{f_1}, \dots, X_{f_n}) = \bar{P}$)
- not same as involutivity!!
$[P, P] \subset P$

A cx. pol is <u>Kähler</u> if $\dim(P \cap \bar{P} \cap TM) = 0$

Ex: (M, ω) Kähler, then
$$T_{\mathbb{C}} M = T^{1,0}M \oplus T^{0,1}M$$

$T^{1,0}M$ - holom. pol. $\qquad$ $T^{0,1}$ - anti-holom. pol.

Prop: if a symp. mfd admits a Kähler pol., then (M, ω) is Kähler

Ex: $S^1 \times S^1 \cong \mathbb{C}/\Lambda$
$S^2 \cong \mathbb{CP}^1$ } admit a Kähler pol.

Rem: not all symp mfds admit a cx polarization ... (constructed via symplectic blowups) counterexamples

Polarized sections:

def (M,ω) symp mfd $(E \to M, \nabla)$ prequantum line bundle
P - cx polarization. A section s of E is called polarized, if
$\nabla_{\bar{X}} s = 0 \quad \forall X$ tangent to P.

Hilbert space: $\mathcal{H}_P$ = the L^2 closure of the space of all polarized sections.

Problems with this definition:

* when the leaves are compact, might happen that $\mathcal{H}_P = 0$
* if leaves are non-compact, many sections but none of them are in L^2
* quantum observables may not preserve the condition of being a polarized section.

$$\begin{array}{ccc} Q(f): \mathcal{H}^{pre} & \longrightarrow & \mathcal{H}^{pre} \\ \cup & & \cup \\ \mathcal{H}_P & \not\longrightarrow & \mathcal{H}_P \end{array}$$

in Kähler case, f must be s.t. X_f is Killing.
- for T^*M, f must be at most linear in f.

Talk III

S is polarized if

$\nabla_{\bar{X}} S = 0$ for X tangent to P (polarization)

Ex: S^2 (with radius 1)

U_S, U_N (stereographic proj.)

$z = \frac{x^1 + i x^2}{1 - x^3}$ in U_S

$\omega_\hbar = i\hbar n \frac{dz \wedge d\bar{z}}{(1+z\bar{z})^2} = 2n\hbar \underbrace{dA}$ the standard area form on S^2

$dA = \frac{i}{2} \frac{dz \wedge d\bar{z}}{(1+z\bar{z})^2}$

$\hbar$-integrality condition

$\left\{ \frac{1}{2\pi\hbar} \int_{S^2} \omega \in \mathbb{Z} \right\}$

$\Theta = -in \frac{\bar{z} dz}{1 + z\bar{z}}$

$K = n\hbar \log(1 + z\bar{z})$

Kähler potential

$\omega = \partial\bar{\partial} K$

$\Theta = -i \partial K$

$S_\psi \leftrightarrow \psi(z) e^{-\frac{K}{2\hbar}}$

$\uparrow$ holom. function

$\left\{ \begin{array}{l} P = \mathrm{span}\{\partial_{\bar{z}}\} \\ \uparrow \text{holom. polarization} \end{array} \right\}$

$\langle S_\psi, S_{\psi'} \rangle = \frac{i}{2\pi} \int_{S^2} \frac{dz \wedge dz'}{(1+z\bar{z})^2} \psi(z) \psi'(\bar{z}) e^{-\frac{K}{\hbar}}$

$= \frac{i}{2\pi} \int_{S^2} \frac{dz \wedge dz'}{(1+z\bar{z})^{2+n}} \psi(z) \psi'(\bar{z})$

$z = Re^{i\Theta}$

$dA = \frac{R\, dR \wedge d\Theta}{(1+R^2)^2}$

$$S_{z^m} = z^m e^{-} \frac{k}{2\pi}$$