Cohomology operations, Steenrod algebra and its dual.

Idea: The more structure the functors $H_*(-)$, $H^*(-)$ have, the more powerful they are in distinguishing spaces.

E.g. • Cup products on $H^*(-)$.

• When $G$ a Lie group ($H$-group). $H_*(G)$, $H^*(G)$ Hopf algebra

Steenrod algebra $A_p$ is an algebra, such that for any $X \in Top$. $H^*(X, \mathbb{Z}/p)$ is an $A_p$-mod, and any map $X \xrightarrow{f} Y$, $f^*$ a $A_p$-mod morphism.

Lead to Adams' solution of vector fields on $S^n$, Lie group ($H$-group) structure on $S^n$, $\mathbb{R}$-division algebra.

Cohomology operations.

Def: $G, K \in Ab$. A cohomology operation of type $(G, m, K, n)$ is a natural transformation $H^m(-, G) \Rightarrow H^n(-, K)$: $h_0(Top) \to Sets$.

Reml: $H^m(-, G)$ considered functor : $h_0(Top) \to Sets$ instead of to $Ab$ because when $G = R$ a ring, we what $(\;)^2 : H^n(-, R) \to H^{2n}(-, R)$ to be a cohomology operation, which is generally not additive.

But interesting operations, the stable ones, are additive Like $\beta$, Bockstein operation

If $0 \to G \to K \to L \to 0$ s.e.s of Ab, since $\mathrm{Hom}_{\mathbb{Z}}(S_n(X), -)$ exact ($S_n(X)$ free), we have l.e.s. which gives $\beta : H^n(X, L) \to H^{n+1}(X, G)$

~~Rem 2: Interesting~~

Classifycation:

Recall $H^i(-, G) \cong [-, K(G, i)]$, then $\{\text{all cohomology operation of type } (G, m, K, n)\} \cong \mathrm{Nat}(H^m(-, G), H^n(-, K)) \cong \mathrm{Nat}([-, K(G, m)], [-, K(K, n)]) \cong [K(G, m), K(K, n)] \cong H^n(K(G, m), K))$

Particularly interesting case: $G = K = \mathbb{Z}/p$. $p$ prime. $H^*(K(\mathbb{Z}/p, n), \mathbb{Z}/p)$ computed by H. Cartan, Serre

Rem: Interesting cohomology operations will increase degree, i.e. $n \geq m$. Since.

Thm (Hurewicz) $X$ connected. Let $\pi_n(X)$ be the first non-trivial homotopy group, $n \geq 2$. Then $H_i(X) = 0$, $i < n$ and $\pi_n(X) \cong H_n(X)$. (When $n = 1$, $\pi_1(X)/[\pi_1, \pi_1] \cong H_1(X)$)

So for $X = K(G, m)$. $\pi_m(X) \cong H_m(X)$, and ~~$H^*(X)$~~ $H^m(X; K)$ is the first non-trivial cohomology group by universal coefficient thm.

When $n=m$, $H^m(K(G,m),K) \cong [K(G,m), K(K,m)]$
$\cong \mathrm{Hom}_{\mathbb{Z}}(G,K)$
Type $(G,m,K,m)$ are coefficient operations.

E.g. (Eilenberg-Maclane spaces) $S^1 \simeq K(\mathbb{Z},1)$. $\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2,1)$
$\mathbb{C}P^\infty \simeq K(\mathbb{Z},2)$

Fix $G=K=\mathbb{Z}/p$ from now on.
Recall the natural isomorphism $H^n(X) \overset{\delta}{\cong} H^{n+1}(\Sigma X)$.

Def A stable cohomology operation of degree $q$ is a sequence of cohomology operations $\theta = \{\theta^n\}$,
$\theta^n : H^n(-,\mathbb{Z}/p) \to H^{n+q}(-,\mathbb{Z}/p)$, s.t. for all $X \in \mathrm{Top}$

$$\begin{array}{ccc} H^n(X,\mathbb{Z}/p) & \xrightarrow{\theta^n} & H^{n+q}(X,\mathbb{Z}/p) \\ \delta\downarrow & & \downarrow\delta \\ H^{n+1}(\Sigma X,\mathbb{Z}/p) & \xrightarrow{\theta^{n+1}} & H^{n+q+1}(\Sigma X,\mathbb{Z}/p) \end{array}$$

Lem Stable operations are additive, i.e., for $x,y \in H^n(X,\mathbb{Z}/p)$
$\theta(x+y) = \theta(x)+\theta(y)$.

Let $A^q = \{\text{all deg } q \text{ stable operations}\}$ $A^0 = \mathrm{Hom}(\mathbb{Z}/p,\mathbb{Z}/p) \cong \mathbb{Z}/p$.
Let $A_p = \bigoplus_{i\geq 0} A^i$, the obvious composition $A^p \otimes A^q \to A^{p+q}$
makes $A$ into a graded $\mathbb{Z}/p$-Algebra.

Thm (Steenrod) For $p=2$. there exists degree $i$ stable operations $Sq^i$, $i \geq 0$, generating $A$.
For $p$ odd prime, there exists degree $2i(p-1)$ stable operations $P^i$, together with $\beta$, generating $A$.

For simplicity, we focus on $p=2$.

Let $R \subseteq A$ be the relation ideal:

$R = \{\theta \in A \mid \theta=0 : H^n(X, \mathbb{Z}/2) \to H^{n+|\theta|}(X, \mathbb{Z}/2)$ for all $X \in Top$ and all $n \geq 0\}$.

~~Def The stee~~

Thm (Adem)

R is generated by

$$\begin{cases} Sq^0 = 1 \\ Sq^a Sq^b = \sum_{c=0}^{[\frac{a}{2}]} \binom{b-c-1}{a-2c}_2 Sq^{a+b-c} Sq^c, \text{ for } a<2b \end{cases}$$

Def The mod 2 Steenrod algebra $A_2$ is the free graded algebra generated by $Sq^i$, quotient the relation $R$. ($|Sq^i| = i$)
More concretely, let $M = \bigoplus_{i \geq 0} \mathbb{Z}/2 \{Sq^i\}$. Then $A_2 = T(M)/R$, $R$ generated by

$$\begin{cases} Sq^0 = 1 \\ Sq^a \otimes Sq^b = \sum_{c=0}^{[\frac{a}{2}]} \binom{b-c-1}{a-2c}_2 Sq^{a+b-c} \otimes Sq^c, \quad a<2b \end{cases}$$

Every $H^*(X, \mathbb{Z}/p)$ is a $A_2$-mod in an obvious way, for $f: X \to Y$, $f^*: H^*(Y, \mathbb{Z}/p) \to H^*(X, \mathbb{Z}/p)$ is a $A_2$-mod morphism.

Thm $Sq^i$ satisfies and is uniquely determined by the axioms

(1) $Sq^i : H^n(X, \mathbb{Z}/2) \to H^{n+i}(X, \mathbb{Z}/2)$ is a natural transformation for all $n$.

(2) $Sq^0 = 1$

(3) $Sq^n(x) = x^2$ if $|x| = n$.

(4) for $|x| < i$, $Sq^i(x) = 0$

(5) Cartan formula

$$Sq^k(x \cdot y) = \sum_{i=0}^{k} Sq^i(x) \cdot Sq^{k-i}(y). \quad (\text{cup product})$$

(6) $Sq^1 = \beta$, the Bockstein of the s.e.s

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

(7) Adem relations

$$Sq^a Sq^b = \sum_{c=0}^{[\frac{a}{2}]} \binom{b-c-1}{a-2c}_2 Sq^{a+b-c} \cdot Sq^c, \quad 0 < a < 2b$$

The $A_2$-mod structure on $H^*(-, \mathbb{Z}/2)$ impose strong constrains:

Thm: If $X$ any topological space, with $H^*(X, \mathbb{Z}/2) = \mathbb{Z}/2[x]$ or $\mathbb{Z}/2[x]/\langle x^m \rangle$, $m \geq 3$. Then $|x| = 2^k$ for some $k$.

Lem: For $m, n \in \mathbb{Z}$, $m = m_0 + m_1 \cdot 2 + \cdots + m_k \cdot 2^k$, $n = n_0 + n_1 \cdot 2 + \cdots + n_k \cdot 2^k$ be 2-adic expansion, then

$$\binom{m}{n} \equiv \binom{m_0}{n_0}\binom{m_1}{n_1} \cdots \binom{m_k}{n_k} \mod 2$$

Def: In a graded algebra $B$. $x \in B$ is called decomposible

If $x = \sum x_i' \cdot x_i''$ with $|x_i'|, |x_i''| < |x|$.

Lem: In $A_2$, if $i \neq 2^k$ for some $k$, $Sq^i$ is decomposible

proof: let $i = i_0 + 2 \cdot i_1 + \cdots + 2^s \cdot i_s$, $i_s \neq 0$

Note the first term in Adem relation is $\binom{b-1}{a}_2 Sq^{a+b}$

We solve $\begin{cases} a+b = i \\ \binom{b-1}{a}_2 = 1 \end{cases}$

let $b = 2^s$, $a = i - b$

$b-1 = 1 + 2 + \cdots + 2^{s-1}$

$a = i_0 + 2 \cdot i_1 + \cdots + 2^{s-1} \cdot i_{s-1}$

then $\binom{b-1}{a}_2 = \binom{1}{i_0}_2 \binom{1}{i_1}_2 \cdots \binom{1}{i_{s-1}}_2 = 1$

So we have

$Sq^a Sq^b = Sq^i + \{\text{products of lower terms}\}$

Proof of thm: Suppose $|x| = i$, then $Sq^i(x) = x^2 \neq 0$

If $i \neq 2^k$ for some $k$. $Sq^i$ is decomposible.

$Sq^i = \sum Sq^{m_j} Sq^{n_j}$, $m_j, n_j < i$

Since $Sq^{n_j}(x) \in H^{i+n_j}(X) = 0$

So $Sq^i(x) = 0$, contradiction. #

Rem: Using secondary cohomology operations, Adams proved $|\alpha|$ can only be 1, 2, 4, 8, ~~implying~~ corresponding to

$\alpha = 1 \quad X = \mathbb{R}P^n$

$\alpha = 2 \quad X = \mathbb{C}P^n \qquad n \geq 1$

$\alpha = 4 \quad X = \mathbb{H}P^n$

$\alpha = 8 \quad X = \mathbb{O}P^n$

With many important corollaries:

Thm: The only division $\mathbb{R}$-algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

$\Rightarrow$ Thm: The only H-~~groups~~ space among $S^n$ are $S^0, S^1, S^3, S^7$

Thm: The maximal number of nowhere linearly dependent vector fields on $S^{n-1}$ is $\rho(n)-1$. (exactly)
($\rho(n)$-Radon-Hurewicz number, $\rho(\text{odd}) = 1$, $\rho(1) = 1$, $\rho(2) = 2$, $\rho(4) = 4$, $\rho(8) = 8$).

$\Rightarrow$

Thm: The only paralizable spheres are $S^0, S^1, S^3, S^7$

Thm: The only Lie groups among $S^n$ are $S^0, S^1, S^3$

Thm: $M^{2n}$ connected compact, $H^n(M;\mathbb{Z}/2) = \mathbb{Z}/2$, $H^i(M;\mathbb{Z}/2) = 0$, $0 < i < n$. Then $|n| = 2^k$ for some $k \leq 3$

proof. using Poincare duality, and Universal coefficients

| | | | |
|---|---|---|---|
| $H^{2n}$ | $\mathbb{Z}/2$ | $H_{2n}$ | $\mathbb{Z}/2$ |
| $\vdots$ | $0$ | $\vdots$ | $0$ |
| $H^n$ | $\mathbb{Z}/2$ | $H_n$ | $\mathbb{Z}/2$ |
| $\vdots$ | $0$ | $\vdots$ | $0$ |
| $H^0$ | $\mathbb{Z}/2$ | $H_0$ | $\mathbb{Z}/2$ |

If $H^n(M) = \mathbb{Z}/2[\alpha]$. Then $\langle \alpha \cup \alpha, [M] \rangle$
$= \langle \alpha, [M] \cap \alpha \rangle = 1 \Rightarrow \alpha^2 \neq 0 \Rightarrow n = 2^k$
for some $k \leq 3$

$H^* = \mathbb{Z}/2[\alpha]/\langle \alpha^3 \rangle$

Additive basis of $A_2$

$I = (i_1, i_2, \dots, i_n, 0, 0, 0 \dots)$ sequence of non-negative integers.

$I$ admissible : $i_k \geq 2 i_{k+1}$

Notation : $Sq^I = Sq^{i_1} Sq^{i_2} \dots Sq^{i_n}$,

If $I = (0, 0, 0 \dots)$, $Sq^I = Sq^0 = 1$

Thm (H. Cartan) Admissible monomials of $Sq^I$ form an additive basis of $A_2$

Proof: Induction using Adem relations.

E.g. Basis of $(A_2)_7$ : $Sq^7$, $Sq^6 Sq^1$, $Sq^5 Sq^2$, $Sq^4 Sq^2 Sq^1$

~~Thm (Serre)~~

For admissible $I$, the excess

$$m(I) = \sum_k (i_k - 2 i_{k-1}) \geq 0.$$

Thm (Serre) $H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2) = \mathbb{Z}/2[Sq^I(\iota)]$, $\iota \in H^n(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$ the fundamental class, $I$ run over all admissible sequences with $m(I) < n$.

E.g. $n=1$. $\mathbb{RP}^\infty \simeq K(\mathbb{Z}/2, 1)$. Only $I = (0, 0, \dots)$ has $m(I) < 1$. $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[\iota]$

Hopf algebras, Steenrod algebra and its dual.

Assume: $K$ a field, all $K$-modules are $\mathbb{N}$-graded $K$-modules of finite type: i.e. $M = \bigoplus_{i\geq 0} M^i$, $M^i$ finitely generated for each $i$.

For $M, N \in K$-mod. A morphism $f: M \to N$ is a $K$-linear map s.t $f(M_n) \subset N_n$.

$(M \otimes N)_n = \bigoplus_{i+j=n} M_i \otimes N_j \qquad (\otimes = \otimes_K)$

We have $(M\otimes N)^* \cong M^* \otimes N^*$ by:

$$(M\otimes N)^*_n = \mathrm{Hom}((M\otimes N)_n, K) = \mathrm{Hom}(\bigoplus_{i+j=n} M_i \otimes N_j, K)$$
$$\overset{f.g.}{\cong} \bigoplus_{i+j=n} \mathrm{Hom}(M_i \otimes N_j, K) = \bigoplus_{i+j=n} M_i^* \otimes N_j^* = (M^*\otimes N^*)_n$$

And $M^{**} \cong M$

Algebras: $(A, \varphi, \eta)$

Def: $A \in K$-mod, $A$ is a $K$-algebra if there are morphisms $A\otimes A \xrightarrow{\varphi} A$, $K \xrightarrow{\eta} A$, s.t.

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "1\otimes\varphi"] \arrow[d, "\varphi\otimes 1"'] & A\otimes A \arrow[d, "\varphi"] \\ A\otimes A \arrow[r, "\varphi"] & A \end{tikzcd} \qquad \begin{tikzcd} K\otimes A \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & A\otimes A \arrow[d, "\varphi"] & A\otimes K \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

A morphism of $K$-algebras $f: A \to B$ is a morphism of $K$-mod and

$$\begin{tikzcd} A\otimes A \arrow[r, "f\otimes f"] \arrow[d, "\varphi_A"'] & B\otimes B \arrow[d, "\varphi_B"] \\ A \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "f"] \\ K \arrow[ur, "\eta_A"] \arrow[r, "\eta_B"'] & B \end{tikzcd}$$

An augmentation of a $K$-algebra $A$ is an algebra morphism $\varepsilon: A \to K$

Coalgebras: $(C, \Delta, \varepsilon)$

Def: $C \in K$-mod is a $K$-coalgebra if there are morphisms $\Delta: C \to C \otimes C$ (comultiplication)

$\varepsilon: C \to K$ (counit), s.t.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes 1"] \\ C \otimes C \arrow[r, "1 \otimes \Delta"] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} K \otimes C & C \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "\Delta"] & C \otimes K \\ & C \otimes C \arrow[ul, "\varepsilon \otimes 1"] \arrow[ur, "1 \otimes \varepsilon"'] & \end{tikzcd}$$

A morphism of $K$-coalgebras $g: C \to D$ is a morphism of $K$-mod and

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "\Delta"] & D \arrow[d, "\Delta"] \\ C \otimes C \arrow[r, "g \otimes g"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[dr, "\varepsilon_C"] \arrow[dd, "g"'] & \\ & K \\ D \arrow[ur, "\varepsilon_D"'] & \end{tikzcd}$$

A coaugmentation of a $K$-coalgebra $C$ is a morphism of $K$-coalgebras $\eta: K \to C$

Bialgebras: $(A, \varphi, \eta, \Delta, \varepsilon)$

A $K$-bialgebra $A$ is a $K$-module with maps

$\varphi: A \otimes A \to A$, $\eta: K \to A$

$\Delta: A \to A \otimes A$, $\varepsilon: A \to K$, s.t.

(1) $(A, \varphi, \eta)$ is an $K$-algebra with augmentation $\varepsilon$.

(2) $(A, \Delta, \varepsilon)$ is an $K$-coalgebra with coaugmentation $\eta$

(3) $\varphi$ is a morphism of $k$-coalgebras, or equivalently, $\Delta$ is a morphism of $k$-algebras:

$$\begin{tikzcd} A\otimes A \arrow[rr, "\varphi"] \arrow[d, "\Delta\otimes\Delta"'] & & A \arrow[dd, "\Delta"] \\ A\otimes A\otimes A\otimes A \arrow[d, "1\otimes T\otimes 1"'] & & \\ A\otimes A\otimes A\otimes A \arrow[rr, "\varphi\otimes\varphi"] & & A\otimes A \end{tikzcd}$$

E.g. (1) $k[x]$ can be made into a bialgebra by defining

$\Delta(x) := 1\otimes x + x\otimes 1$

$\Delta(x^n) = (\Delta(x))^n = (1\otimes x + x\otimes 1)^n$

If char $k=2$, or $|x|$ even.

$\Delta(x^n) = 1\otimes x^n + x^n\otimes 1 + \sum_{i=1}^{n-1} \binom{n}{i} x^i \otimes x^{n-i}$

$\varepsilon, \eta$ are obvious ones.

(2) $\Lambda_k[x]$, $|x|$ odd.

($0 = \Delta(x^2) = (\Delta(x))^2 = x\otimes x + (-1)^{|x|^2} x\otimes x$, if $|x|$ even, then $\Delta$ is not an algebra morphism unless char $k=2$)

(3) $G$, connected lie group, with $H_*(G,k)$, $H^*(G,k)$ of finite type.

$\times$ : $H_*(G)\otimes H_*(G) \cong H_*(G\times G) \xrightarrow{m_*} H_*(G)$

$\Delta$ : $H_*(G) \xrightarrow{\Delta_*} H_*(G\times G) \cong H_*(G)\otimes H_*(G)$

$\cup$ : $H^*(G)\otimes H^*(G) \cong H^*(G\times G) \xrightarrow{\Delta^*} H^*(G)$

$\Delta$ : $H^*(G) \xrightarrow{m^*} H^*(G\times G) \cong H^*(G)\otimes H^*(G)$

Hopf algebras $(A, \varphi, \eta, \Delta, \varepsilon, S)$

A Hopf algebra $A$ is a $k$-bialgebra with a $k$-linear map $S: A \to A$, s.t.

$$\begin{tikzcd} & A\otimes A \arrow[r, "1\otimes S"] & A\otimes A \arrow[dr, "\varphi"] & \\ A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[r, "\varepsilon"] & A \arrow[rr, "\eta"] & & A \\ & A\otimes A \arrow[r, "S\otimes 1"] & A\otimes A \arrow[ur, "\varphi"'] & \end{tikzcd}$$

E.g. (1) $K[x]$, $\Lambda_K[x]$, $S(x) = -x$

(2) $G$ connected Lie groups, $S: H_*(G) \to H_*(G)$ or $H^*(G) \to H^*(G)$ induced by the inverse $i: G \to G$

Thm (Milnor) $A_2$ is a Hopf algebra over $\mathbb{Z}/2$, with $\Delta(Sq^n) = \sum_{i=0}^{n} Sq^i \otimes Sq^{n-i}$, $S(Sq^n) = Sq^n + \sum_{i=1}^{n-1} S(Sq^i)\cdot Sq^{n-i}$ defined inductively. $\varepsilon, \eta$ are the obvious ones.

Let $I_k = (2^{k-1}, 2^{k-2}, \ldots, 2, 1)$. $\xi_k \in A_2^*$ the dual basis of $Sq^{I_k}$

$A_2^* = \mathbb{Z}/2[\xi_1, \xi_2, \ldots]$

Thm (Milnor) $A_2^*$ is a Hopf algebra over $\mathbb{Z}/2$, $|\xi_i| = 2^i - 1$

$\Delta(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$,

$S(\xi_k) = \xi_k + \sum_{i=1}^{k-1} [S(\xi_{k-i})]^{2^i} \cdot \xi_i$

defined inductively, $\varepsilon, \eta$ are the obvious ones.