Connor | **topological K-theory**

$BU(n) = \{ \text{cx } n\text{-planes in } \mathbb{C}^\infty \}$

$BU = \underrightarrow{\text{colim}}\ BU(n)$

$BU(n) \hookrightarrow BU(n+1)$

$\mathbb{C} \oplus \mathbb{C} \oplus \dots \to \mathbb{C} \oplus \mathbb{C} \oplus$ shift

$(x_1, x_2, \dots) \longmapsto (0, x_1, x_2, \dots)$

- char. class ω is stable if $\omega(\xi \oplus \underbrace{[1]}_{\text{trivial bdle}}) = \omega(\xi)$

stable $\rightsquigarrow$ if we mod out by trivial bdles, the char. class is still defined

reduced stable vect. bundles of X are $\text{Vect}(X) / \text{reduced stable equivalence}$ "$RSV(X)$"

$\xi \sim \xi'$ iff $\exists\, [K], [K']$ trivial v.b. s.t. $\xi \oplus K = \xi' \oplus K'$

recall $\forall \xi\ \exists\, \xi^c$ "complement" characterized by $\xi \oplus \xi^c = [n]$

For X connected compact, $RSV(-) \cong [-, BU]$ functor

- RSV is not good enough!

why? - it is only representable for connected spaces
- we loose the notion of rank of a bundle

- **def** If M is commutative monoid, then $Gr(M)$ is the group $\varphi : M \longrightarrow Gr(M)$

Grothendieck group of M

so $\forall f : M \to G$ where G is an ab. grp, we have

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varphi"'] & G \\ Gr(M) \arrow[ur, dashed, "\exists!\, \tilde{f}"'] & \end{tikzcd}$$

Talk | Recall |

Grothendieck group of a monoid $Gr(M)$ satisfies

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\varphi"'] & G \\ Gr(M) \arrow[ur, dashed, "\exists ! f"'] & \end{tikzcd}$$

Construction ①

$$Gr(M) = \bigoplus_{m \in M} \mathbb{Z}\overline{m} \Big/ \left(\overline{m_1} + \overline{m_2} = \overline{m_1 + m_2} \right)$$ – very large!

Construction ②

$$M \oplus M /_{\sim} \quad = \quad M \oplus M / \Delta$$

quotient by smallest submonoid containing the diagonal, giving an equiv. relation

$(a,b) \sim (c,d) \iff \exists k$ so that $a+d+k = c+b+k$.

$(a,b) =$ "$a-b$"

X - compact

$K(X) = Gr(Vect(X))$

(reduced) cohom. theory in general:

$\overline{h}^{*} : CW_{+}^{op} \to Ab.Grps$ $\quad * \in \mathbb{Z}$ s.t.

1) $\overline{h}^{*}$ homotopy-invariant

2) $\overline{h}^{k}(\vee X_i) = \prod \overline{h}^{k}(X_i)$

3) (X, A) a CW pair, we have a natural LES

$$\cdots \leftarrow \overline{h}^{n+1}(X/A) \xleftarrow{\delta} \overline{h}^{n}(A) \xleftarrow{i^*} \overline{h}^{n}(X) \xleftarrow{q^*} \overline{h}^{n}(X/A) \xleftarrow{\delta} \cdots$$

Reduced K-theory:

$\overline{K}(X) \subseteq K(X)$ with the property $a-b \in \overline{K}(X)$ iff $\operatorname{rank} a - \operatorname{rank} b = 0$ over the base point

subset

virtual dimension

(might be $\neq 0$ over other conn. components of X)

For a connected space, BU still represents $\overline{K}(-)$

$$X \rightleftarrows *$$

$$\text{coker} \leftarrow K(X) \leftarrow K(*)$$

$$\text{coker} = BSV$$

$$0 \leftarrow BSV(X) \leftarrow K(X) \leftarrow K(*) \leftarrow 0$$

$$K(X) \cong K(*) \oplus BSV(X)$$

$$\ker \to K(X) \to K(*)$$

$$\ker = \tilde{K}(X)$$

$$BSV(X) \cong \tilde{K}(X)$$

vect bundles / trivial vect. bundles

"BSV" = R⊃V

this implies that BU represents $\tilde{K}(X)$ for a connected compact X.

If I have a group-like H-space B, i.e. $\pi_0(B)$ forms a group

then: path component of 1 $\hookrightarrow B \to B/$

- $\tilde{K}(S^0) = \mathbb{Z}$

$[n]-[n] = 0$

$[n]-[m] = [n-m]$ if $n > m$

associate $[m-n]$

$\to$ analysis gives us: represented by $BU \times \mathbb{Z}$

$(x,n) \in BU \times \mathbb{Z}$

(x,n) thought of as a virtual vector bundle of local rank n.

$X \hookrightarrow BU \times \mathbb{Z}$ - can construct a virtual v.b. from such a map. over X

$\xi_x \times [n - \mathrm{rk}\, \xi]$ ← triv bdle

Any h^* has a suspension iso $\bar{h}^n(X) \cong \bar{h}^{n+1}(\Sigma X)$

$\bar{h}^*$ is known $\Rightarrow$ $\bar{h}^{-*}$ is known

$\bar{h}^{-k}(X) = \bar{h}^0(\Sigma^k X)$

Bott periodicity: $\Omega^2(BU \times \mathbb{Z}) = BU \times \mathbb{Z}$

(based) double loop space

$h^* \Leftrightarrow$ sequence of loop spaces X_i where we choose $\Omega X_{i+1} = X_i$

$\ldots, \Omega(BU \times \mathbb{Z}), BU \times \mathbb{Z}, \Omega(BU \times \mathbb{Z}), \ldots$ – alternating sequence

$\Omega(BU \times \mathbb{Z}) = \Omega BU$

for a top group G, $\boxed{\Omega BG \simeq G}$

why?

fiber over basepoint

$$\begin{tikzcd} G \arrow[r] \arrow[d] & EG \arrow[r] \arrow[d] & BG \arrow[d, "="] \\ \Omega BG \arrow[r] & PBG \arrow[r] & BG \end{tikzcd}$$

path space fibration

recall: EG is contractible, so pick a contraction

$EG \ni x \to$ to the path it takes under contraction

apply π_*:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_k(G) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \pi_k(BG) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \pi_k(\Omega BG) \arrow[r] & 0 \arrow[r] & \pi_k(BG) \arrow[r] & \cdots \end{tikzcd}$$

by 5-lemma, this gives a weak equiv. $G \simeq \Omega BG$

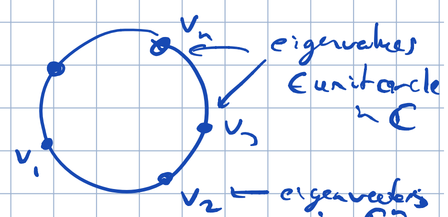
Sketch of the rest of Bott periodicity (proof by Mark Behrens)

recall

$U(n)$ – set of unitary operators $\mathbb{C}^n$

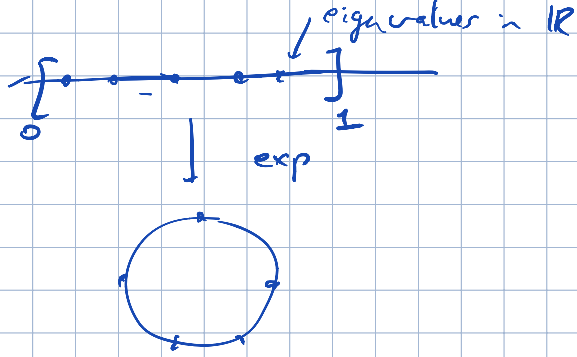
$H(n)$ – hermitian operators

$U(n)$:



$v_1 \oplus \cdots \oplus v_n$ is an orthog decomp of $\mathbb{C}^n$

$H(n)$:



$U(n)$

$H \xrightarrow{\exp} U$ $\big(BU\times\mathbb{Z} \to A \xrightarrow{\exp} U\big)$

$\Omega U \to PU \to U$ is a quasi-fibration

equiv. to path space fibration

H
$\cup$
U
$\cup$
A
$=$
colim Hermitian operators with eigenvalues lying in $[0,1]$

Talk III

Bott periodicity

says we can define $K^n(X) = \begin{cases} [X, BU\times\mathbb{Z}] & \text{if } n \text{ even} \\ [X, U] & \text{if } n \text{ odd} \end{cases}$

This makes $K^\bullet$ a coho theory

e.g. $\tilde{K}^n(S^{2k}) = \mathbb{Z}$ if n even, $= 0$ if n odd | $\tilde{K}^n(S^{2k-1}) = \mathbb{Z}$ if n odd, 0 if n even

reduced (base-pointed maps)

Relative K-theory

Say, if (X, A) an "adequate pair",

$K^0(X,A) = \{ (\xi, \xi', \alpha) \mid \xi, \xi'$ are complex v.b. over X, α is an iso $\xi|_A \to \xi'|_A \} / \sim$

$K^0(X, *) \ni (\xi, \xi')$ s.t. $\operatorname{rank} \xi|_* = \operatorname{rank} \xi'|_*$.

Let E be a cohomology theory

V - real v.b. ~~X~~ that is "E-oriented"

Thom isomorphism theorem: if X is compact, then $E^\bullet(X) = E^{\bullet+n}(\mathrm{Tot}(V), \mathrm{Tot}(V) - X)$ (zero section)

If X not compact,

$$\bar{E}^*(X^\bullet) = E^{*}(\mathrm{Tot}(V), \mathrm{Tot}(V) - X)$$

1-pt compactification

Thom construction:

If X is compact, $Th(V) = \mathrm{Tot}(V)^\bullet$

If X non compact, first compactify each fiber, then identify the added infinite points.

Fact: $\bar{E}^*(Th(V)) \cong E^*(\mathrm{Tot}(V), \mathrm{Tot}(V) - X)$

If $E = K$, then any cx v.b. is K-orientable.

Recall: if φ is an elliptic differential operator

$$\Gamma(\underset{\text{tot space}}{\xi : T} \to X) \longrightarrow \Gamma(\underset{\text{tot space}}{\xi' : T'} \to X)$$

$\pi : T^*X \to X$

$S = \mathrm{Symbol}(\varphi)$ is a map $\pi^*(\xi) \to \pi^*(\xi')$

S is an isomorphism away from $X \subseteq T^*X$

$\boxed{(\pi^*(\xi), \pi^*(\xi'), S)} \in K(T^*X, T^*X - X)$ — a rel. K-theory class

X is a compact manifold

$X \overset{\text{embed}}{\hookrightarrow} S^{2n-1}$ Codim $= \ell$ with normal bundle μ

$T^*X \hookrightarrow T^*S^{2n-1}$ codim $= 2\ell$

T^*X [illegible]

$K(T^*X, T^*X - X) = \bar{K}(Th(T^*X)) = \bar{K}((T^*X)^\bullet)$ compactify fiber

// Thom iso

$K^{2\ell}(\mathrm{Tot}(\mu), \mathrm{Tot}(\mu) - X)$

//

$K^{2\ell}(Th(\mu))$

// Bott perio.

$K(Th(\mu))$

$(T^*S^n)^\bullet \to \mathrm{Tot}(\mu)^\bullet$ from $\mathrm{Tot}(\mu) \to T^*S^n$

Note: $\mu \oplus \mu$ K-orientable!

$Tot(\mu)^\circ \longrightarrow Th(\mu)$ - from the univ property of 1-pt compactification

$\tilde{K}(Th(\mu)) \longrightarrow \tilde{K}(Tot(\mu)^\circ) \longrightarrow \tilde{K}(Th(T^* S^n))$ ⊜

$\Sigma \, Th(T^* S^n) = S^{2n+1} \vee S^{n+1}$

$Th(T^* S^n)$ has cohomology equal to $S^{2n} \vee S^n$

⊜ $\mathbb{Z} \oplus 0 = \mathbb{Z}$

composition of these maps, evaluated on the symbol S
is the topological index of φ.

[ref: Stephan's RTG video]

$X \hookrightarrow S^n \hookrightarrow S^{n+2}$
induces same homom. via Bott periodicity.

embeddings $X \hookrightarrow S^n$ for N large are isotopic

$\rightsquigarrow$ t-ind is well-defined!

AS index theorem:

a-ind(φ) = t-ind(φ)