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SW classes
Grassmanian

Stiefel-Whitney classes

colom - with coeff in $\mathbb{Z}_2$
bundle maps - injective on fibers

given $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ rk $= n$ v.bun., want $w(E) \in H^*(B)$

$$w(E) = w_0 \in H^0(B) + w_1 \in H^1(B) + \cdots$$

such that

- $\forall E \quad w_0(E) = 1$ and $w_k(E) = 0, \quad k > \operatorname{rk} E$
- $f^* w(E) = w(f^* E), \quad f: B' \to B$ (natural)
- $w(E \oplus F) = w(E) \cup w(F)$
- $w_1(\gamma_1^1) = a \neq 0 \in H^1(S^1)$

taut bun. over $\mathbb{R}P^1 \simeq S^1$ — generator of $\mathbb{Z}_2$ — so, $w(\gamma_1^1) = 1 + a$

Thm: these exist

If M, M' are two n-dim. closed mfds, then M, M' are in the same cobordism class $\Leftrightarrow$ SW numbers? of TM, TM' are the same.

Grassmanians

paracompact, Hausdorff Top spaces / homotopy equiv. $\xrightarrow{F}$ Sets

$B \longmapsto \{$isom. classes of rk $= n$ vector bundles$\}$

$$F \simeq \operatorname{Hom}(-, G_n)$$

G_n — Grassmanian

$G_n(\mathbb{R}^m)$

$$V_n^0(\mathbb{R}^m) = \{(v_1, \dots, v_n) \subset (\mathbb{R}^m)^{\times n} \mid \{v_i\} \text{ are orthonormal}\}$$

$O(n)$ acts on $V_n^0(\mathbb{R}^m)$

$$G_n(\mathbb{R}^m) = V_n^0(\mathbb{R}^m) / O(n)$$

so, $V_n^\circ(\mathbb{R}^m) = \left\{\begin{array}{l} n\text{-planes in } \mathbb{R}^m \\ \text{with o.n. basis} \end{array}\right\}$

$$\downarrow$$

$G_n(\mathbb{R}^m) = \{n\text{-planes in } \mathbb{R}^m\}$

$$V_n^\circ(\mathbb{R}^m) \times_{O(n)} \mathbb{R}^n =: \gamma^n_{m-n}$$

$$\downarrow \pi$$

$$G_n(\mathbb{R}^m)$$

$\pi^{-1}(p) = \{\text{all vectors in said plane } p\}$

Lemma for a v.b. $E \to B$ ← cpt, Hausdorff, $\exists$ a map $E \to \gamma_k^{\text{rank } E}$ for $k \gg 0$

Proof

$U_1 \ldots U_r$ – cover; $\{\lambda_i\}$ – partition of unity w/ supp $\lambda_i \subset U_i$
trivializations

$$E \longrightarrow \mathbb{R}^{n \times r}$$

$$(b, v) \longmapsto (\lambda_1(b)v, \ldots, \lambda_r(b)v)$$

...

$G_n = \varinjlim_m G_n(\mathbb{R}^m)$

$\gamma^n = \varinjlim_m \gamma^n_{m-n}$

$G_n(\mathbb{R}^m) \longrightarrow G_n(\mathbb{R}^{m+1})$
induced by $\mathbb{R}^m \hookrightarrow \mathbb{R}^{m+1}$
also maps induced by

$$\begin{array}{ccc} \mathbb{R}^m \times \ldots \times \mathbb{R}^m \\ \downarrow \quad\quad \downarrow \\ \mathbb{R}^\infty \times \ldots \times \mathbb{R}^\infty \end{array}$$

Thm for B paracompact Hausdorff, $E \to B$ vec. bun., $\text{rk} = n$

$\exists f: E \to \gamma^n$

furthermore $f, g: E \to \gamma^n$ are homotopic

Proof existence is similar, if $\forall e \in E_{\neq 0}$ $f(e) \neq -g(e)$

$\mathbb{R}^\infty \to \mathbb{R}^\infty$

i^{th} coord $\to 2i^{\text{th}}$ coord

$\to 2i-1$ st coord

$\boxtimes$

$\rightsquigarrow$ we have a bijection between $\left\{\begin{array}{l} \text{iso classes of} \\ \text{rk} = n \text{ v.b.}/B \end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{l} \text{homotopy classes} \\ \text{of maps } B \to G_n \end{array}\right\}$

back to SW classes

$w(E)$:

$E \xrightarrow{f} \gamma^n$

$f_w^*(\gamma^n) =: w(E)$ — so, we get SW classes by pulling back cohom. classes of the Grassmanian !!!

CW structure on G_n

$G_n(\mathbb{R}^m)$, for P an n-plane in $\mathbb{R}^m$, let $v_1(p) \dots v_n(p)$ be a basis

$$\begin{bmatrix} v_1(p) \\ \vdots \\ v_n(p) \end{bmatrix} \xrightarrow{\text{RREF}} \left[\underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{n} \middle| \underbrace{\begin{matrix} * & * & * & * \\ - & - & - & - \\ * & * & * & * \end{matrix}}_{m-n}\right] \simeq \mathbb{R}^{(m-n)n} \underset{\text{homeo}}{\simeq} D^{(m-n)n}$$ — top-cell in $G_n(\mathbb{R}^m)$

$\mathbb{R}P^2$ $\quad [x \ y \ z] \xrightarrow{\text{RREF}} [1 \ y \ z]$ general position 2-cell

special case

$[0 \ y \ z] \longrightarrow [0 \ 1 \ z]$ 1-cell

special case

$[0 \ 0 \ 1]$ 0-cell

attaching maps are $\underline{\times 2}$

- $\# \{r\text{-cells in } G_n(\mathbb{R}^m)\}$
$= \# \{\text{partitions of } r \text{ into } n \text{ integers between } 0 \text{ and } m-n\}$

- $Sk_r\, G_n(\mathbb{R}^m) \xrightarrow{\sim} Sk_r G_n(\mathbb{R}^{m+1})$ $\quad m-n \geq r$

$\rightsquigarrow$ direct limit G_n is a CW complex.

$H^*(G_n, \mathbb{Z}_2)$

$$\begin{array}{ccc} \mathbb{R}P^\infty \times \dots \times \mathbb{R}P^\infty & & \\ \uparrow & & \\ \gamma^1 \boxplus \dots \boxplus \gamma^1 & \xrightarrow[f]{} & \gamma^n \\ e^{(1)} \cdot \ldots \cdot e^{(n)} & & \end{array}$$

$f^* w_k(\gamma^n) = w_k(\gamma^1 \boxtimes \dots \boxtimes \gamma^1)$