The Atiyah-Singer index theorem (Updated talk)

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1 References.

- Levi Lopes de Lima, "The index formula for Dirac operators: an introduction".
- Friedrich Hirzebruch, Matthias Kreck, "On the concept of genus in topology and complex analysis".
- Rafe Mazzeo, "The Atiyah-Singer index theorem: what it is and why you should care".
- Liviu Nicolaescu, "Notes on the Atiyah-Singer index theorem".
- John Roe, "Elliptic operators, topology and asymptotic methods".

2 Introduction.

Consider the following theorems:

1. Riemann-Roch. $l(D) - l(K - D) = \deg D + 1 - g(X)$, where K is a canonical divisor and D is any divisor on a Riemann surface X, and $l(D)$ is the dimension of the vector space of meromorphic functions f on X whose divisor (f) makes $(f)+D$ effective. The LHS contains analytic information, while the RHS contains topological information in $g(X)$.

2. Hirzebruch signature theorem. $\text{sign}(M) = L[M]$ is the L-genus of M. One can think of the LHS as containing analytic information via Hodge theory, while the RHS contains topological information.

3. Chern-Gauss-Bonnet. $\chi(M) = \int_M e(\Omega)$. The LHS contains topological information in the Euler characteristic of M, and the RHS contains analytic information. (Alternatively, one can compute the LHS by computing the index of some operator and the RHS topologically, as we will soon see.)

4. $\chi(V) = \int_V T_n(V)$ is the Todd genus of V, when V is a nonsingular compact complex algebraic variety of dimension n. The LHS contains analytic information in the holomorphic Euler number, and the RHS contains topological information.

These theorems relate analytic and topological information. They generalize to the Atiyah-Singer index theorem.

3 The Atiyah-Singer index theorem.

The Atiyah-Singer index theorem computes the index of some operator in terms of topological invariants. The general form of the theorem applies to elliptic pseudodifferential operators. Its statement and proof involve K-theory, which Connor will talk about. Here we look at the special case that applies to twisted Dirac operators.

3.1 The set-up.

(See Ch. 8 of de Lima.) Let us consider a spin manifold M of dimension $n = 2k$. It comes with a canonical bundle called the spinor bundle $\mathcal{S}(M)$ with connection $\nabla^{\mathcal{S}}$. There is the Atiyah-Singer-Dirac operator $\partial\!\!\!/$ on $\mathcal{S}(M)$, defined by

$$\partial\!\!\!/ : \mathcal{C}^\infty(\mathcal{S}(M)) \xrightarrow{\nabla^{\mathcal{S}}} \mathcal{C}^\infty(T^*M \otimes \mathcal{S}(M)) \to \mathcal{C}^\infty(\mathcal{S}(M)).$$

In terms of the local frame for TM, the operator is $\partial\!\!\!/ = \sum_{i=1}^n e_i \cdot \nabla^{\mathcal{S}}_{e_i}$, so we see that it is a first-order linear differential operator. It is formally self-adjoint, hence has index 0. Clearly we are not computing $\operatorname{ind}(\partial\!\!\!/)$ in the Atiyah-Singer index theorem.

$\mathcal{S}(M)$ has a decomposition $\mathcal{S}(M) = \mathcal{S}^+(M) \oplus \mathcal{S}^-(M)$ that respects the metric and $\nabla^{\mathcal{S}}$, such that $\partial\!\!\!/(\mathcal{C}^\infty(\mathcal{S}^\pm(M))) \subseteq \mathcal{C}^\infty(\mathcal{S}^\mp(M))$. We denote $\partial\!\!\!/^\pm = \partial\!\!\!/ \mid_{\mathcal{C}^\infty(\mathcal{S}^\pm(M))}$. These operators $\partial\!\!\!/^\pm$ are formal adjoints, so $\operatorname{ind}(\partial\!\!\!/^+) = \dim\ker \partial\!\!\!/^+ - \dim\ker \partial\!\!\!/^-$. This is the index we want to compute.

Now we give a slightly more general set-up, involving twisted Dirac operators. Say we have a hermitian vector bundle $\mathcal{G}$ over M with compatible connection $\nabla^{\mathcal{G}}$, then the bundle $\mathcal{S}(M) \otimes \mathcal{G}$ inherits the decomposition $\mathcal{S}(M) \otimes \mathcal{G} = \mathcal{S}^+(M) \otimes \mathcal{G} \oplus \mathcal{S}^-(M) \otimes \mathcal{G}$. There is the twisted Dirac operator $\partial\!\!\!/_{\mathcal{G}}$ defined by

$$\partial\!\!\!/_{\mathcal{G}} : \mathcal{C}^\infty(\mathcal{S}(M) \otimes \mathcal{G}) \xrightarrow{\nabla^{\mathcal{S}(M)\otimes\mathcal{G}}} \mathcal{C}^\infty(T^*M \otimes \mathcal{S}(M) \otimes \mathcal{G}) \to \mathcal{C}^\infty(\mathcal{S}(M) \otimes \mathcal{G}).$$

As above, we can define $\partial\!\!\!/^\pm_{\mathcal{G}}$, which are adjoints. In this case we compute $\operatorname{ind}(\partial\!\!\!/^+_{\mathcal{G}}) = \dim\ker \partial\!\!\!/^+_{\mathcal{G}} - \dim\ker \partial\!\!\!/^-_{\mathcal{G}}$.

3.2 The theorem.

Theorem. *(Atiyah-Singer for twisted Dirac operators) In the above set-up,*

$$\mathit{ind}(\partial\!\!\!/^+_{\mathcal{G}}) = \int_M \hat{A}(TM) \wedge \mathit{ch}(\mathcal{G}),$$

where $ch(\mathcal{G})$ denotes the Chern character of $\mathcal{G}$.

The Chern character of a bundle is constructed from its Chern classes, so it is a topological invariant of the bundle. (More explanation will follow.) So the LHS contains analytic information, and the RHS is computed by topological invariants.

In particular, if M has dimension $n = 4l$, and we let $\mathcal{G} = \underline{\mathbb{C}}$, then we get the index of the Atiyah-Singer-Dirac operator $\partial\!\!\!/^+$:

Theorem. *(Atiyah-Singer for Atiyah-Singer-Dirac operators)*

$$\mathit{ind}(\partial\!\!\!/^+) = \int_M \hat{A}(TM) = \hat{A}\text{-genus of } M.$$

If M is a 4-dimensional spin manifold, then it follows immediately from the theorem that $\hat{A}(M) = -\frac{1}{24}\int_M p_1(TM)$ is an integer, which is not obvious from the definition of the Pontrjagin classes.

3.3 Chern character.

(See Ch. 7 of de Lima.) The Chern character $\mathrm{ch}(\mathcal{E})$ of a vector bundle $\mathcal{E}$ is made from the Chern classes $c_i(\mathcal{E})$ of $\mathcal{E}$. By the splitting principle, to compute $\mathrm{ch}(\mathcal{E})$, we only need the case when $\mathcal{E} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r$ is a sum of line bundles. Let $x_i = c_1(\mathcal{L}_i)$, then $c(\mathcal{E}) = c(\mathcal{L}_1)\cdots c(\mathcal{L}_r) = (1+x_1)\cdots(1+x_r)$. Expanding the expression so that $c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \ldots$, where $c_k(\mathcal{E}) \in H^{2k}(M;\mathbb{Q})$, we see that $c_k(\mathcal{E}) = \sigma_k(x_1, \ldots, x_r)$ is the k-th elementary symmetric function in x_i.

We define the Chern character to be $\mathrm{ch}(\mathcal{E}) = \sum_i e^{x_i} \in H^*(M;\mathbb{Q})$. Expanding the expression so that $\mathrm{ch}(\mathcal{E}) = \mathrm{ch}_0(\mathcal{E}) + \mathrm{ch}_1(\mathcal{E}) + \ldots$, where $\mathrm{ch}_k(\mathcal{E}) \in H^{2k}(M;\mathbb{Q})$, we get

$$\mathrm{ch}(\mathcal{E}) = r + \underbrace{x_1 + \ldots + x_r}_{c_1(\mathcal{E})} + \underbrace{\frac{1}{2}\left(x_1^2 + \ldots + x_r^2\right)}_{\frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E}))} + \ldots.$$

Note that $\mathrm{ch}(\mathcal{E} \oplus \mathcal{E}') = \mathrm{ch}(\mathcal{E}) + \mathrm{ch}(\mathcal{E}')$ and $\mathrm{ch}(\mathcal{E} \otimes \mathcal{E}') = \mathrm{ch}(\mathcal{E})\mathrm{ch}(\mathcal{E}')$, so it is a ring homomorphism. Alternatively, we can define $\mathrm{ch}(\mathcal{E}) = \left[\mathrm{tr}\left(e^{-\Omega/2\pi i}\right)\right] \in H^*(M;\mathbb{Q})$ where Ω is the curvature form of some connection on $\mathcal{E}$. (This follows the Chern-Weil construction that computes topological invariants from connections and curvatures.)

4 Applications.

4.1 The Chern-Gauss-Bonnet formula.

(See Ch. 10.1 of de Lima.) We use the theorem to prove the special case of the Chern-Gauss-Bonnet formula when the manifold is spin.

Theorem. *(Chern-Gauss-Bonnet) Let M be a spin manifold of dimension $n = 2k$, then its Euler characteristic is $\chi(M) = \int_M e(TM)$, where $e(TM)$ is the Euler class of TM.*

To apply Atiyah-Singer, we shall find some twisted Dirac operator whose index is $\chi(M)$.

Let $\Lambda^{\mathrm{even}}(M)$ denote the bundle of even-degree complex differential forms over M, and let $\mathcal{A}^{\mathrm{even}}(M) = \mathcal{C}^\infty(\Lambda^{\mathrm{even}}(M))$. Define the same things for odd-degree forms.

Consider the operator $\mathcal{D} = d + d^* : \mathcal{A}^{\mathrm{even}}(M) \to \mathcal{A}^{\mathrm{odd}}(M)$. By Hodge-de Rham theory, $\mathrm{ind}(\mathcal{D}) = \chi(M)$. However, this operator $\mathcal{D}$ is not a twisted Dirac operator with respect to the grading $\Lambda(M) = \Lambda^{\mathrm{even}}(M) \oplus \Lambda^{\mathrm{odd}}(M)$. Instead we consider the twisted Dirac operator $\partial\!\!\!/^{+}_{(-1)^k\hat{\mathcal{S}}(M)}$, where $\hat{\mathcal{S}}(M) = \mathcal{S}^+(M) - \mathcal{S}^-(M)$. It turns out that $\mathrm{ind}(\mathcal{D}) = \mathrm{ind}\left(\partial\!\!\!/^{+}_{(-1)^k\hat{\mathcal{S}}(M)}\right)$. Now we just compute the two topological invariants appearing on the RHS of the Atiyah-Singer formula:

1. $\mathrm{ch}(\hat{\mathcal{S}}(M)) = \prod\left(e^{-y_i/2} - e^{+y_i/2}\right) = (-1)^k y_1 \cdots y_k + \text{h.o.t.} = (-1)^k e(TM) + \text{h.o.t.}$, where y_i are some Chern classes of $\mathcal{S}^\pm(M)$. (See details in Ch. 10.1 of de Lima.)

2. $\hat{A}(TM) = 1 + \text{h.o.t.}$.

Therefore,

$$\chi(M) = \mathrm{ind}\left(\partial\!\!\!/^{+}_{(-1)^k\hat{\mathcal{S}}(M)}\right) = \int_M \hat{A}(TM) \wedge \mathrm{ch}((-1)^k\hat{\mathcal{S}}(M)) = \int_M (-1)^{2k} e(TM) = \int_M e(TM).$$

4.2 Topological obstructions of positive scalar curvature.

Lichnerowicz proved that a compact spin manifold with non-zero $\hat{A}$-genus does not admit any metric of strictly positive scalar curvature. Here is a sketch of its proof (See Theorem 13.1 in Roe):

Let $\partial\!\!\!/$ be the Atiyah-Singer-Dirac operator on the spinor bundle. The Weitzenböck formula says that $\partial\!\!\!/^2 = \nabla^*\nabla + \frac{1}{4}\kappa$, where κ is scalar curvature. The Bochner vanishing argument says that if the least eigenvalue of κ is strictly positive, then there are no non-zero solutions of $\partial\!\!\!/^2 s = 0$. (See Theorem 3.10 in Roe.) If there is $\kappa > 0$, then since all eigenvalues are positive, this argument implies that $\ker \partial\!\!\!/ = \ker \partial\!\!\!/^2 = 0$. Then Atiyah-Singer says that $0 = \mathrm{ind}(\partial\!\!\!/^+) = \hat{A}$-genus, a contradiction.

Gromov and Lawson proved that any simply-connected closed *non-spin* manifold of dimension ≥ 5 has a metric of positive scalar curvature. Stolz later proved that any simply-connected closed *spin* manifold of dimension ≥ 5 has a metric of positive scalar curvature if and only if some characteristic number of that manifold is 0.

4.3 Rokhlin's theorem.

By Atiyah-Singer, the $\hat{A}$-genus of a 4-dimensional spin manifold is the index of the Atiyah-Singer-Dirac operator $\partial\!\!\!/$. In this case, the kernel and cokernel of $\partial\!\!\!/$ have a quaternionic structure, so they are even-dimensional as $\mathbb{C}$-vector spaces, hence there is a factor of 2 in the index, and the $\hat{A}$-genus is even. (See Proposition 13.3 in Roe.)

Rokhlin proved that a closed compact oriented simply-connected *smooth* 4-fold with even intersection form has signature that is divisible by 16. This is because $\hat{A}$-genus$= -\frac{1}{8}\ \mathrm{sign}(M)$. M. Freedman later showed the existence of a closed compact oriented simply-connected *topological* 4-fold with even intersection form whose signature is 8. Therefore this manifold has no smooth structure. (See 3.2.36-38 in Nicolaescu.)

4.4 Etc.

Richard will talk about the Hirzebruch signature theorem and Hirzebruch-Riemann-Roch.