Matt | Talk 0

Cobordism.

cpt. smooth

def M^n, N^n smooth mfds

M, N are cobordant :if $\exists W^{n+1}$ s.t. $\partial W = M \amalg N$

can ask for additional structures:

- M, N orientedly cobordant if $\exists W$ s.t. $\partial W = M \amalg (-N)$
  (M, N or, W or)

$\Omega_n^O = (\{n\text{-mfds}\}/\text{cobordism}, \amalg)$ ($\amalg$ = addition)

$\Omega_n^{SO} = (\{\text{oriented } n\text{-mfd}\}/\text{cob.}, \amalg)$

} abelian groups

ring structures $\times : \Omega_n^O \times \Omega_m^O \to \Omega_{n+m}^O$, $M, N \mapsto M \times N$

(graded-commutative)

Prop Ω_*^O is an $\mathbb{F}_2$-algebra

$M \amalg M = \partial(M \times [0,1]) = [\emptyset]$

Lem: $[M] = [\emptyset]$

$\Leftrightarrow \exists W^{n+1}$ st. $M = \partial W$.

Talk 2

def The i-th Pontryagin class of ξ (a vect. bun.) is

$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(M, \mathbb{Q})$

($c_{2i+1}(\xi \otimes_{\mathbb{R}} \mathbb{C})$ is 2-torsion (due to conjugation) $\Rightarrow$ vanishes rationally)

$n = \dim M$

$H^*(BSO(n), \mathbb{Q}) = \mathbb{Q}[p_1, \dots, p_k, \chi(?)]$ (Euler class) $\quad k = \left[\frac{n}{4}\right]$

($BSO(n)$: oriented Grassmanian)

$BSO(n) \hookrightarrow BSO(n+1)$

$\mathbb{R}^\infty \hookrightarrow \mathbb{R}^\infty$ (shift by 1 coordinate)

$\gamma \in BSO \longmapsto \text{shift}(\gamma) \oplus \text{Span}(e_1)$

$BSO = \lim\limits_{\longrightarrow} BSO(n)$

trv. bundle

def A characteristic class w is stable if $w(\xi \oplus \varepsilon) = w(\xi)$

Fact: $H^*(BSO;\mathbb{Q}) = \{\text{stable char. classes}\}$

$= \mathbb{Q}[p_1, p_2, \ldots]$

Ex: w_i, c_i, p_i

Non-example: χ - Euler class

Def: Let $v \in H^n(M;\mathbb{Q})$ $\quad v[M] \in \mathbb{Q}$ ($[M]$: orientation class)

Let w a char. class. The assoc. characteristic number is $w(\tau_M)[M] =: w[M]$ (τ_M: tangent bundle)

Ex: Pontryagin #'s $\boxed{p_1^{i_1} \cdots p_k^{i_k}[M]}$ - can be nonzero only if dim M divisible by 4.

Thm (Wall)
M oriented mfld $M = \partial W$ iff all Pontryagin and Stiefel-Whitney numbers vanish.

Thm (Thom) Pontryagin #'s $= 0 \iff \exists k \quad k[M] = 0$

Thm (Thom) SW # $= 0 \iff M = \partial W$ as unoriented mflds.

Pf Assume $[M] = [\emptyset]$. (one direction)

① $M = \partial W$

by collared nbhd thm $\rightarrow$ $TW|_M = TM \oplus \varepsilon$

stability $i^* p(TW) = p(TM)$

②
$$\partial : H_{n+1}(W, M) \longrightarrow H_n(M)$$
$$\partial : [W] \longmapsto [M]$$
$$\delta : H^n(M) \longrightarrow H^{n+1}(W, M)$$
$$p[M] = \pm \delta p[W]$$

③
$$H^n(W) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(M, M)$$
exact

$$p[M] = \delta i^* p(\tau_W)[W]$$
$\quad = 0$

∎

Also works with Pontryagin $\longrightarrow$ SW + unoriented

Construction 1 Let $\xi \to M$ a v.b. Call $Sph(\xi) :=$ fiberwise pt compactification

The **Thom space** $T\xi := Sph(\xi) / e_\infty$ ← added pts at ∞.

$$H^*(T\xi, \infty) = H^*_{cv}(\xi)$$

(∞: rel. to the pt at ∞; cv: compactly supported in vert. direction.)

Cons. 2 Let $M^k \hookrightarrow \mathbb{R}^N$ w/ normal bundle ν.

Then we have a map $S^N \longrightarrow T\nu$

$N_\varepsilon(M) \to \nu$

everything else $\longmapsto \infty$

So have a map $\nu \to \gamma_{N-k}$ ← classifying bundle over BSO.

$$S^N \to T\nu \longrightarrow T\gamma_{N-k}$$

Fact: $(N \gg k)$ this is invariant of embedding up to homotopy
"   cobordism class

def A **prespectrum** is a sequence of spaces E_n w/ maps

$\sigma_n : \Sigma E_n \to E_{n+1}$

Ex: $MSO_k := T\gamma_k$

σ_k

$$\begin{array}{ccc} \gamma_k \oplus \varepsilon & \longrightarrow & \gamma_{k+1} \\ \downarrow & & \downarrow \\ BSO(k) & \longrightarrow & BSO(k+1) \end{array}$$

$$T(\gamma_k \oplus \varepsilon) \xrightarrow{\sigma_k} T\gamma_{k+1}$$
$$\cong$$
$$\Sigma T\gamma_k$$

def $\pi_n(E_*) = \varinjlim_k \pi_{n+k}(E_k)$

$$H_n(E_*) = \varinjlim_k H_{n+k}(E_k)$$

Thm (Pontryagin - Thom)

Cons. 2 gives a map $\alpha : \Omega_n^{SO} \to \pi_n(MSO)$

In fact, this is an isomorphism.

Pf: transversality...

Proof of Thom thm:

$$\begin{array}{ccccc} H^n(BSO)\otimes\Omega_n^{SO}\otimes\mathbb{Q} & \xrightarrow{\alpha} & H^n(BSO)\otimes\pi_n(MSO)\otimes\mathbb{Q} & \xrightarrow[\text{Hurewicz}]{h_{\mathbb{Q}}} & H^n(BSO)\otimes H_n(MSO) \\ \downarrow & & & & \downarrow \Phi \ \text{Thom iso} \\ \mathbb{Q} & & \xleftarrow{\langle , \rangle} & & H^n(BSO)\otimes H_n(BSO) \end{array}$$

$v\otimes[M]\otimes 1 \mapsto v(T_M)[M]$

Fact: - this commutes

Fact: Φ is an iso

Pf: Thom iso + induction

Fact: $h_{\mathbb{Q}}$ is an iso

Pf: Thm (Rat. Hurewicz) X simply connected, $\pi_i(X)$ finite for $i\le n$

$\pi_i(X)\otimes\mathbb{Q} \xrightarrow{\sim} H_i(X,\mathbb{Q})$ is an iso $i\le 2n$.

Assume Pontryagin #'s $=0$. Then $w[M]=0$ $\forall w\in H^n(BSO)$

$\to \langle w, \Phi\circ h_{\mathbb{Q}}\circ\alpha[M]\rangle = 0 \quad \forall w$

$\Rightarrow \Phi\circ h_{\mathbb{Q}}\circ\alpha[M]=0$

$\Rightarrow [M]\otimes 1 = 0$

$[M]$ is torsion

$\square$

Side effect: $\Omega_*^{SO}\otimes\mathbb{Q} \cong H_*(BSO) \cong \mathbb{Q}[p_1, p_2, \ldots]$

Fact: $\cong \mathbb{Q}[\mathbb{C}P^{2i}]$

Talk 3

Multiplicative genera

def a multiplicative genus is a ring map $\Omega^G \to \Lambda$

Rem: for this talk, $\Omega^G = \Omega^{SO}$, $\Lambda = \mathbb{Q}$

(so, in fact $\Omega^{SO} \otimes \mathbb{Q}$)

Ex: $\Omega^O_* \to \mathbb{F}_2 := (M \to \chi(M) \bmod 2)$

- $\chi(M \sqcup M') = \chi(M) + \chi(M')$
- $\chi(M \times M') = \chi(M)\chi(M')$
- $\chi(\partial M) \equiv 0 \pmod 2$

Ex: If M^{4n} oriented $\exists$ non-deg. symm. biln. form $H^{2n}(M, \mathbb{R})$ given by Poincaré duality.

$\sigma(M) =$ signature of this
$\#\{\text{positive eigenval}\} - \#\{\text{negative e.v.}\}$

– this is a multiplicative genus.

def let A be a comm, graded ring

$A^\pi := \{1 + a_1 + a_2 + \dots \mid a_i \in A_i\}$ – group

Let $\{K_n\}$ a seq. of degree $= n$ homog. polynomials in n variables x_i where $|x_i| = i$.

Given such $a \in A^\pi$ write $K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \dots$

we say $\{K_n\}$ a mult. seq. if $K(ab) = K(a)K(b)$

Ex: $1 + \lambda a_1 + \lambda^2 a_2 + \lambda^3 a_3 + \dots$

$\lambda \in \mathbb{Q}$ $\quad K_n = \lambda^n a_n$ – multiplicative sequence

def: Given mult seq. $\{K_n\}$, the K-genus of an oriented M^{4n}:

$$K[M] := K_n(p_1(\tau_M), \dots, p_n(\tau_M))[M]$$

Pontryagin classes of the tangent bundle

Lemma (Hirzebruch)

$$\left\{ \begin{matrix} \text{mult.} \\ \text{seq.} \end{matrix} \right\} \longleftrightarrow \mathbb{Q}[[z]]$$

via given $\{K_n\}$ mult. $K(1+z) =: f(z)$ is a power series

Thm $S_n \circlearrowright \mathbb{Q}[t_1, \dots, t_n]$ $|t_i| = 1$

$$\mathbb{Q}[t_1, \dots, t_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n]$$

$|\sigma_i| = i$
↑ elem. sym. polynomials

Given $f = \sum \lambda_i z^i$

write $\prod_{j=1}^{n} f(t_j z) = \sum_{\ell} z^{\ell} \underbrace{\sum_{|I|=\ell} t_1^{i_1} \cdots t_n^{i_n} \lambda_{i_1} \cdots \lambda_{i_n}}_{*}$

$*$ symmetric

If $n \geq \ell$, write

$K_\ell(\sigma_1, \dots, \sigma_\ell) =: *$

$\{K_\ell\}$ is mult. seq.! Proof in Milnor-Stasheff.

Ex: L-genus is the genus assoc. to

$$\frac{\sqrt{z}}{\tanh\sqrt{z}} = \sum_k \frac{2^{2k} B_{2k} t^k}{(2k)!}$$

Rem: $$\frac{t e^{xt}}{e^t - 1} = \sum_n B_n(x) \frac{t^n}{n!}$$

$$B_n = B_n(0)$$

Rem: $$\int_0^k B_n(t)\,dt = \sum_{i=0}^{k} i^n$$

Thm (Hirzebruch)

$$L[M] = \sigma[M]$$

Proof: Atiyah-Singer (or check on $\mathbb{C}P^{2n}$)

- can use also to prove num. theoretic properties of Pontryagin numbers.

Def the $\hat{A}$-genus is assoc. to

$$\frac{\frac{\sqrt{z}}{2}}{\sinh \frac{\sqrt{z}}{2}}$$

Fact (A-S) M spin $\hat{A}[M] \in \mathbb{Z}$

$-8 \hat{A}_1 = L_1$

M^4 spin $\rightsquigarrow$ $\sigma(M) \in 16\mathbb{Z}$

def The Todd genus is $\Omega^U \longrightarrow \mathbb{Q}$

$\uparrow$ a cobordism

$= \mathbb{Z}[\mathbb{C}P^n]$ Pf: Adams Spectral Seq.

assoc. to $\frac{z}{1-e^{-z}}$ using Chern classes

Fact: V cx var $T[V]$ = arithmetic genus.