Elliptic Operators and Analytic Index

Definition: Let $\Omega \subseteq \mathbb{R}^n$ be an open domain. A second order differential operator L given by

$$Lu = a_{ij}D^{ij}(u) + b_i D^i(u) + c(u)$$

is said to be elliptic (on Ω) if for all $\xi \in \mathbb{R}^n$, there exist $\lambda, \Lambda > 0$ such that

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for all $x \in \Omega$. Equivalently, L is elliptic if the matrix a_{ij} is positive. If $\frac{\lambda}{\Lambda}$ is bounded, then L is said to be uniformly elliptic.

More generally, a differential operator of order k, which has the form

$$Lu = \sum_{|\alpha| \leq k} a_\alpha D^\alpha u$$

is said to be elliptic if for all $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$ we have that

$$\sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha \neq 0$$

For a differential operator L, we can define the symbol of L to be the polynomial

$$p_L(x, \xi) = \sum_{|\alpha| \leq k} a_\alpha(x)\xi^\alpha$$

where $\xi = (\xi^1, \ldots, \xi^k)$. The principal symbol of L is the truncated polynomial

$$\sigma_L(\xi)(x) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$$

Example: The most classical example of an elliptic operator is the Laplacian Δ, which is defined by

$$\Delta u = \nabla \cdot \nabla u = D^{jj}u$$

Of course here $a_{ij} = \delta_{ij}$, so it is easy to see that Δ is uniformly elliptic.

Definition: Let M be a smooth compact manifold. We say that $L : C^\infty(M) \to C^\infty(M)$ is a differential operator (on M) if L is of the form above in any chart U_α of M. Similarly, L is said to be elliptic if it satisfies the properties for ellipticity locally.

Remark: Of course, a differential operator L of order k can be considered as an operator from either $C^\infty(M)$ into itself or as an operator from $C^k(M)$ into $C(M)$.

Example: Suppose that (M, g) is a Riemannian manifold. The Laplace-Beltrami operator Δ_g is given by

$$\Delta_g\varphi = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial\varphi}{\partial x^j}\right)$$

Note that if $M = \mathbb{R}^n$ and g is the standard Euclidean metric, then Δ_g agrees with our definition of Δ. Furthermore, one can check that $\Delta_g \varphi = \operatorname{div} \operatorname{grad} \varphi$ for suitable definitions of divergence and gradient on a Riemannian manifold, as is the case in $\mathbb{R}^n$.

Definition: We can define an inner product on $C^\infty(M)$ in the usual way via

$$\langle u, v \rangle = \int_M u\bar{v} dx$$

Here, dx is a smooth Borel measure on M (it is locally the Lebesgue measure). For a differential operator L, we can then define the adjoint of L in the usual way by requiring that $L^* : C^\infty(M) \to C^\infty(M)$ satisfy

$$\langle Lu, v \rangle = \langle u, L^* v \rangle$$

for all $u, v \in C^\infty(M)$.

Theorem (Fredholm): If M is compact and L is an elliptic operator on M, then $\ker L$ is finite-dimensional and $C^\infty(M) \ni u \in \operatorname{ran} L$ if and only if $\langle u, v \rangle = 0$ for all $v \in \ker L^*$.

This theorem is useful because it allows us to talk about the analytic indices of elliptic differential operators on compact smooth manifolds.

Definition: Let L be an elliptic operator on a compact smooth manifold M. The (analytic) index of L is defined to be

$$\operatorname{ind} L = \dim \ker L - \dim \ker L^*$$

Example: Let $M = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and let $L = \frac{d}{dx} - \lambda$ for some $\lambda \in \mathbb{R}$. Elements of $\ker L$ are of the form $e^{\lambda x}$ if λ is an integer multiple of $2\pi i$ and zero otherwise. A simple computation shows that elements of $\ker L^*$ are then of the form $e^{\bar{\lambda} x}$ if $\lambda \in 2\pi i \mathbb{Z}$ and 0 otherwise, so L has analytic index 0.

Elliptic Complexes

Definition: Let M be a compact smooth manifold as before. Let E^k be a rank-k vector bundle over M. Let $d : C^\infty(E^k) \to C^\infty(E^{k+1})$ be a first-order differential operator. We say that the complex

$$C^\infty(M) \xrightarrow{d} C^\infty(E^1) \xrightarrow{d} \ldots \xrightarrow{d} C^\infty(E^k) \xrightarrow{d} \ldots$$

is a differential complex if $d^2 = 0$.

For $x \in M$ and $\xi \in T_x^* M$, we can define $\sigma_\xi : F_x^k \to F_x^{k+1}$ on the fibers of the E^k's in a way that is analogous to the case for $\mathbb{R}^n$ and then define $\sigma_d(x, \xi)$ to be the symbol of d by

$$\sigma_d(x, \xi) e = d(gs)(x)$$

where $g \in C^\infty(M)$ such that $dg_x = \xi, g(x) = 0$ and $s \in \Gamma(M, E^k)$ such that $s(x) = e$. We see that

$$\sigma_{d^2}(x, \xi) = \sigma_d(x, \xi) \sigma_d(x, \xi) = 0$$

which implies that $\sigma_\xi^2 = 0$. This gives us a complex

$$0 \xrightarrow{\sigma_\xi} E_x^1 \xrightarrow{\sigma_\xi} E_x^2 \xrightarrow{\sigma_\xi} \dots$$

We say that our differential complex is an elliptic complex if the complex we get on the fibers is exact for every $x \in M$ and every $\xi \in T_x^* M$.

Example (de Rham Complex): For a smooth compact manifold M, let $E^k = \Omega^k(M)$ - the space of differential k-forms on M. Let $d : \Omega^k(M) \to \Omega^{k+1}(M)$ be the exterior derivative defined in the usual way: for $\omega = f dx^1 \wedge \dots \wedge dx^k$,

$$d\omega = \frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge \dots \wedge dx^k = \frac{\partial f}{\partial x^{k+1}} dx^1 \wedge \dots \wedge dx^{k+1}$$

One can check that in this case, $\sigma_\xi(d)(x, \xi)$ becomes $\cdot \wedge \xi$. One can then use this to check that the complex given by d is elliptic.

References:

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