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Equivariant cohomology

Borel approach

$X \circlearrowleft G$
mfd group

$X \xrightarrow{f} Y$ G-equivariant map
if $f(X^g) = f(X)^g$

homotopy of G-equiv-maps
$h: X \times I \longrightarrow Y$

- homotopy through G-equiv. maps

$G_2 \circlearrowright S^\infty \longrightarrow RP^\infty$

antipodal action

S^∞ non-equivariantly contractible!

we would like to say $H^*_G(X) \overset{?}{=} H^*(X/G)$

if X not free, X/G can be terrible

homotopy quotient: $X_G := (X \times EG)/G$

Borel equivariant cohomology: $\boxed{H^*_G(X) := H^*(X_G)}$

Lemma: $f: X \to Y$ is a non-equivar. homotopy equivalence and is G-equivariant then $H^*_G(X) \cong H^*_G(Y)$

Pf

$$\begin{array}{ccc} G & = & G \\ \downarrow & & \downarrow \\ EG \times X & \xrightarrow{1 \times f} & EG \times Y \\ \downarrow & & \downarrow \\ X_G & \xrightarrow{(1\times f)_G} & Y_G \end{array}$$

use the induced LES on cohomology and 5-lemma

Mayer-Vietoris $M = U_1 \cup U_2$, have M-V seq.

G-invariant subsets

also: Serre spectral sequence

$$\begin{array}{c} M \\ \downarrow \\ M_G \\ \downarrow \\ BG \end{array}$$

$$E_2^{p,q} = H^p(BG) \otimes H^q(M) \Rightarrow H^*(M_G)$$

Ex: ① X contractible $X \to *$

$$H^*_G(X) = H^*_G(*) = H^*(BG)$$ - group cohomology.

② $X \circlearrowleft G$ free $\quad X_G \underset{he}{\simeq} X/G \quad H^*_G(X) = H^*(\underbrace{X/G}_{\text{orbit space}})$

③ X, G acts trivially. $\rightsquigarrow X_G = BG \times X$

$H^*_G(X) \cong H^*(BG) \otimes H^*(X)$ [with coeff in a field; otherwise can be a Tor/Ext correction term from Künneth f-la]

④

$S^1 \circlearrowleft \underbrace{S^2}_{D^+ \cup D^-}$

two disks

$A = D^+ \cap D^- \simeq S^1$

reduced cohom $\tilde{H}^*_G = 0$

$$H^*_G(D) = H^*(BS^1) = \mathbb{Z}[x^2]$$

degree 2 generator

$$H^*_G(S^2, \mathbb{Z}) = \mathbb{Z}[x_+^2, x_-^2] \Big/ (x_+^2 x_-^2 = 0)$$

- from M-V

Talk 2
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$H_G(X) = H^*(EG \times X/G)$ Borel equiv. cohomology.

what is the analog of diff forms story?

M . G-mfd $\qquad \varphi_g : M \xrightarrow{\text{diffeo}} M$; $g \in G$ cpt Lie

- $\mathfrak{g} \to \Gamma(TM)$ $\qquad \xi \longmapsto a_\xi = \frac{d}{dt}\Big|_{t=0} \varphi_{\exp(-t\xi)}$
- $G \circlearrowright \Omega(M)$ $\qquad \rho_g \omega = \varphi_{g^{-1}}^* \omega$ ← to make it a left action
- $\mathfrak{g} \circlearrowright \Omega(M)$ $\qquad \mathcal{L}_\xi = \frac{d}{dt}\Big|_{t=0} \rho_{\exp(t\xi)}$

Fact: $\mathcal{L}_\xi$ is a derivation on $\Omega(M)$ of degree 0:

- $\mathcal{L}_\xi : \Omega^k \to \Omega^k$, $\qquad \mathcal{L}_\xi(\omega \wedge \eta) = \mathcal{L}_\xi \omega \wedge \eta + \omega \wedge \mathcal{L}_\xi \eta$ Leibniz rule
- d - derivation of deg $+1$.
- ι_ξ - derivation of degree -1.
  contraction with a_ξ

$\mathcal{L}_\xi, d, \iota_\xi$ satisfy "Weil equations"

$d^2 = 0 \qquad d\iota_\xi + \iota_\xi d = \mathcal{L}_\xi$

$\iota_\xi \iota_\eta = -\iota_\eta \iota_\xi \qquad d\mathcal{L}_\xi = \mathcal{L}_\xi d$

$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]} \qquad \mathcal{L}_\xi \iota_\eta - \iota_\eta \mathcal{L}_\xi = \iota_{[\xi,\eta]}$

more abstractly:

def a $\mathfrak{g}$-differential algebra is any graded algebra with derivations $d, \mathcal{L}_\xi, \iota_\xi$ satisfying Weil equations.

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if $M \circlearrowleft G$ free, $\qquad H^*_G(M) = H^*(M/G)$
- how to model this on forms?

def $\mathfrak{g}$-horizontal subalgebra:
$\mathcal{A}_{horiz} = \bigcap \ker(\iota_\xi)$
$\mathfrak{g}$-invariant subalg.
$\mathcal{A}^{\mathfrak{g}} = \bigcap \ker(\mathcal{L}_\xi)$

basic subalgebra: $\mathcal{A}_{bas} = \mathcal{A}^{\mathfrak{g}}_{horiz}$
- inv. & horizontal forms

Fact $\Omega(M)_{basic} = \Omega(M/G)$ for $M \circlearrowleft G$ free

Fact if ω basic, then $d\omega$ is also basic

so, def: $H^*_{basic}(\mathcal{A}) = H^*(\mathcal{A}_{basic})$.

$G \circlearrowleft M$ is locally free iff given a basis $\{\xi_i\}$ in $\mathfrak{g}$, vect. fields a_{ξ_i} are lin. indep.

$\Rightarrow \exists$ dual $\theta^i \in \Omega^1(M)$

$L_{\xi_i}\theta^j = \delta_i^j$

$L_i := L_{\xi_i}$

θ^i - "connections" (name)

A has connections if $\exists\, \theta^i$ s.t. $L_i\theta^j = \delta_i^j$

$\underset{\text{acyclic}}{E}$ has connections $\theta^i \Rightarrow \mathcal{A}\otimes \underset{\text{acyclic}}{E}$ has connections $1\otimes\theta^i$

$H^*_{basic}(\mathcal{A}\otimes E)$ for E acyclic, has cons

Fact: choice of E is irrelevant. ($\sim$ different models for EG are equivalent)

· first model of E: $\Omega(V_{n,\infty})$

$\underset{\overrightarrow{m}}{\lim}\, \underbrace{V_{n,m}}_{\text{space of } n\text{-frames in } \mathbb{R}^m} =: V_{n,\infty}$ colimit

$\underset{\overleftarrow{m}}{\lim}\, \Omega(V_{n,m}) =: \Omega(V_{n,\infty})$ projective limit

fact: $\Omega(V_{n,\infty})$ is acyclic ($\leftarrow$ lower cohomology of $V_{n,m}$ vanishes)

fact: $\Omega(V_{n,\infty})$ has connections (inherited from cons on $\Omega(V_{n,m})$)

NB! $V_{n,\infty}$ is a model of EGL_n

——— // ——— EG.

fact: $\Omega(M)\otimes\Omega(V_{n,\infty}) \hookrightarrow \Omega(M\times V_{n,\infty})$ induces an iso in basic cohomology.

Thm (equivariant de Rham thm)

$$H^*_G(M,\mathbb{R}) \cong H^*_{basic}(\Omega(M)\otimes\Omega(V_{n,\infty}))$$

$M \circlearrowleft G$ does not have to be free!

pf: $H^*_G(M) = H^*(M_G) = H^\cdot(\Omega(M_G)) = H^\cdot_{basic}(\Omega(M\times V_{n,\infty}))$

$$= H^\cdot_{basic}(\Omega(M)\otimes\underbrace{\Omega(V_{n,\infty})})$$

can be replaced by any acyclic alg. w/ cons

Second model of E

Weil algebra

$$W_{\mathfrak{g}} = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$$

θ^i deg=1, μ^i deg=2 — generators

$$d(\theta^i\otimes 1) = 1\otimes\mu^i$$
$$d(1\otimes\mu^i) = 0$$
$$\iota_a\theta^b = \delta^b_a$$

fact: $W_{\mathfrak{g}}$ is acyclic, has connections $\theta^i\otimes 1$

In fact: if $\mathcal{A}$ is any $\mathfrak{g}$-da w/ cons,

$\exists!\ \rho: W_{\mathfrak{g}} \to \mathcal{A}$ (~ classifying map) which sends $\theta^i\otimes 1$ to cons on $\mathcal{A}$

(so, $W_{\mathfrak{g}}$ is the "smallest" $\mathfrak{g}$-da w/ cons)

fact:

$$W_{basic} = 1\otimes(S\mathfrak{g}^*)^G \longleftrightarrow W_{horiz} = 1\otimes S\mathfrak{g}^*$$

Corollary: $H^\cdot_G(*) = H^\cdot_{basic}(W_{\mathfrak{g}}) = (S\mathfrak{g}^*)^G$ ← THM!!!

$\quad\ \ \| \\ H^\cdot(BG)$

also: ρ gives a map (Chern-Weil)

$$K_G : S(\mathfrak{g}^*)^G \longrightarrow H^*_{basic}(\mathcal{A})$$

or

$$H^*_G(M)$$

Thm (Cartan) $\underbrace{H^*((S\mathfrak{g}^*\otimes\mathcal{A})^G)}_{\text{Cartan model}} \simeq \underbrace{H^\cdot_{basic}(W_{\mathfrak{g}}\otimes\mathcal{A})}_{\text{Weil model}}$

Pf: $(W \otimes A)_{horiz} = W_{hor} \otimes A$ ← Mathai-Quillen isomorphism.