Classifying spaces BG

G - topological group. We focus on based CW complexes.

Thm (Steenrod) Let $\hat{p}: \hat{E} \to \hat{B}$ a principal G-bundle $\hat{\xi}$, Then if $\pi_i(\hat{E}) = 0$ for all $i \geq 0$. $(\hat{E}, \hat{p}, \hat{B})$ is a universal principal bunde, that is,

$$\underset{f}{[-, \hat{B}]} \xrightarrow[T]{\cong} \underset{f^*\xi}{\mathrm{Prin}_G(-)} : \mathrm{ho}(CW_*) \longrightarrow \mathrm{Sets}$$

is a nature equivalence.

proof: We only need to prove when E is n-connected. $[X, \hat{B}] \cong \mathrm{Prin}_G(X)$ for all CW complex X with $\dim X \leq n$.

~~Lem 1.~~

Construction: (associated bundle).

Let $p: E \to B$ be a G-bundle ξ, Y a left G-space

We have an associated bundle $\xi[Y]$

Total space: $E[Y] = E \times Y / (eg, y) \sim (e, gy)$

$$\begin{array}{cc} E[Y] & (e, y) \\ {\scriptstyle p_Y}\downarrow & \downarrow \\ B & p(e). \end{array}$$

Fiber of $E[Y]$ is Y.

Lem 1: $\xi = (E, p, B)$, $\xi' = (E', p', B')$ two G-b

there is one-to-one correspondence b

$\{$bundle map $\xi \to \xi'\} \leftrightarrow \{$ sections of $\xi[E']$

sketch: Given $\phi: \xi \to \xi'$, define

$$\begin{tikzcd} E \times E'/\sim \\ B \arrow[u, "\lambda"'] \end{tikzcd}$$

$\lambda(b) = [e, \phi(e)]$. $e \in p^{-1}(b)$ well-defined

Given $\lambda: B \to E \times E'/\sim$.

consider

$$\begin{tikzcd} E \arrow[r, "\lambda \circ p"] \arrow[d, "p"'] & (E \times E')/\sim \\ B \arrow[ur, "\lambda"'] \end{tikzcd}$$

$\lambda \circ p(e) = (e, \phi(e))$ for same ϕ.

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d] & E' \arrow[d] \\ B \arrow[r, "\Phi"] & B' \end{tikzcd}$$

Φ induced from $E/G \to E'/G$ #

lem 2. Any bundle map of G-bundles

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr] & & E' \arrow[dl] \\ & B \end{tikzcd}$$

over same base space is isomorphism.

sketch: Locally,

$$U \times G \xrightarrow{f} U \times G$$

$f(u, g) = (u, g \cdot h(u))$, $h : u \to G$.
So define $f^{-1}(u,g) = (u, g\cdot h(u)^{-1})$

Globally.

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr] & & E' \arrow[dl] \\ & B & \end{tikzcd}$$

f is a bijection and local homeomorphism, so a global homeomorphism.
#

To prove $T_X : [X, \hat{B}] \to \mathrm{Prin}_G(X)$. surjective, we only need a bundle map $\xi \to \hat{\xi}$. then

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr] \arrow[dd] & & \hat{E} \arrow[dd] \\ & f^*\hat{\xi} \arrow[ur] \arrow[dl] & \\ B \arrow[rr, "f"] & & \hat{B} \end{tikzcd}$$

$\xi \cong f^*\hat{\xi}$. (T surj).
But a map $\xi \to \hat{\xi} \iff$ a section $\xi[\hat{E}]$.
Now we turn it into an extension problem:

Lem 3: (X, A) relative CW, with $\dim(X,A) \leq n$.
$\xi = (E, p, X, F)$ a fiber bundle. with section $\lambda_A : A \to E$. If F $(n-1)$ connected. Then we have an extension of λ_A to a global section $\lambda : X \to E$.

sketch:
$$\begin{tikzcd} E \arrow[r, "p"] & X \arrow[l, dashed, bend left, "\lambda"'] \\ A \arrow[u, "\lambda_A"] \arrow[r, phantom, "\subset"] & X \arrow[u, "1"'] \end{tikzcd}$$
reduce to
$$\begin{tikzcd} E \arrow[r, "p"] & X \\ S^{r-1} \arrow[u] \arrow[r] & D^r \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$
$(r \leq n)$

$F \to E \to X$. F $(n-1)$ connected.

$\pi_n(E) \xrightarrow{P_*} \pi_n(B) \to \pi_{n-1}(F) \to \pi_{n-1}(E) \to \pi_{n-1}(B) \to \cdots$

$\Rightarrow$ $\pi_i(E) \to \pi_i(B)$ iso $i<n$, epic. $i=n$

Replace B by M_p.

$$\begin{tikzcd} & M_p \arrow[d, shift left, "r \simeq"] \\ E \arrow[ur, "j"] \arrow[r, "p"] & B \arrow[u, shift left, "i"] \end{tikzcd}$$

$$\begin{tikzcd} & \pi_n(M_p) \arrow[r] \arrow[d, "\cong"] & \pi_n(M_p, E) \\ \pi_n(E) \arrow[ur] \arrow[r] & \pi_n(B) & \end{tikzcd}$$

$\Rightarrow \pi_i(M_p, E) = 0$. $i \le n$.

$$\begin{tikzcd} E \arrow[r] & M_p \\ S^{r-1} \arrow[u, "\lambda_{S^{r-1}}"] \arrow[r, hook] & D^r \arrow[u, "g"'] \arrow[ul, dashed, "\lambda_{D^r}"] \end{tikzcd} \quad \Leftarrow \quad [D^r, S^{r-1}] \xrightarrow{g} [M_p, E]$$

$r \le n$.

$\pi_r(M_p, E) = 0 \Rightarrow g|_{S^{r-1}} = \lambda_{S^{r-1}}$ trivial

$\Rightarrow \lambda_{D^r}$ exist.

#.

We need a section $\lambda: X \to \xi[E']$.

But fiber of $\xi[\hat{E}'] \to X$ is $\hat{E}'$, n-connected

Let $A = \{x_0\}$.

$$\begin{tikzcd} E \times \hat{E}' / \sim \arrow[r] & X \\ \{x_0\} \arrow[u, "\lambda_{x_0}"] \arrow[r, hook] & X \arrow[u, "1"'] \arrow[ul, dashed, "\hat{\lambda}"] \end{tikzcd}$$

$\lambda_{x_0}(x_0) = (e_0, \hat{e}_0)$

$\Rightarrow T$ surjective.

T: injective: Let $f_0, f_1 \in [X, \hat{B}]$. with $f_0^*\xi \cong f_1^*\xi$.

Denote this total space E.

$$\begin{tikzcd} f_i^*\xi \arrow[r] \arrow[d] & \hat{E} \arrow[d] \\ X \arrow[r, "f_i"] & \hat{B} \end{tikzcd} \quad \longleftrightarrow \text{ sections } \quad \begin{tikzcd} E\times\hat{E}/\sim \\ X \arrow[u, "\lambda_i"] \end{tikzcd}$$

Consider bunde

$$\begin{tikzcd} E\times I \arrow[d] \\ X\times I. \end{tikzcd} \quad \text{with} \quad \begin{tikzcd} E\times\{i\} \arrow[d] \\ X\times\{i\} \end{tikzcd} = f_i^*(\hat{\xi})$$

We have sections

$$(E\times I)[\hat{E}] \quad \begin{tikzcd} E[\hat{E}]\times I \arrow[r] & X\times I \arrow[dl, dashed, "\lambda"'] \\ X\times \dot{I} \arrow[u] \arrow[r] & X\times I \arrow[u] \end{tikzcd} \quad \lambda \longleftrightarrow \quad \begin{tikzcd} E\times I \arrow[r, "H"] \arrow[d] & \hat{E} \arrow[d] \\ X\times I \arrow[r, "\bar{H}"] & \hat{B} \end{tikzcd}$$

with $\dim(X\times I, X\times\dot{I}) \leq n+1$.

$\bar{H}_0 = f_0$, $\bar{H}_1 = f_1 \Rightarrow f_0 \simeq f_1$

#.

Milnor construction

Topological joint: X, Y topological space.

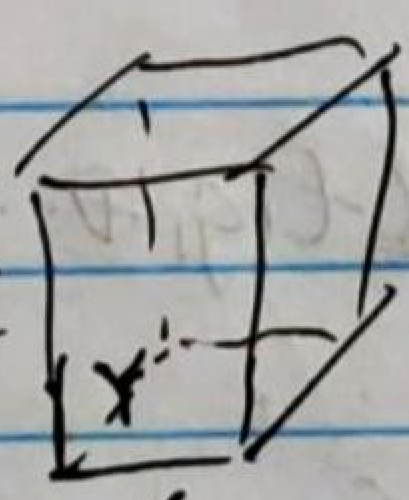
$X * Y = X\times I\times Y/\sim$

$(x, 0, y) \sim (x, 0, y_1)$, $(x, 1, y) \sim (x_1, 1, y)$

for all x, x_1, y, y_1

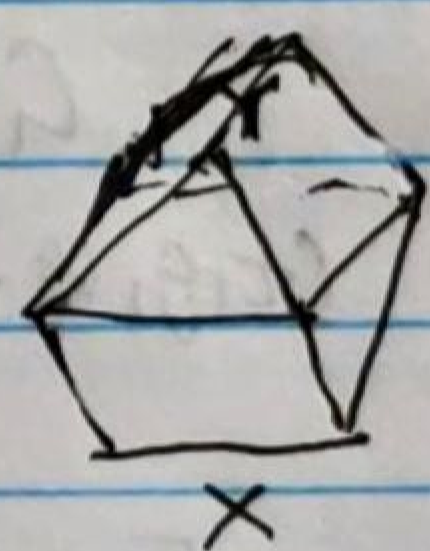
$X\times I\times Y$

I



X

$X * Y$



X

e.g: $X * \{pt\} = X\times I\times\{pt\}/ X\times\{1\}\times\{pt\} = CX$

Lem: $X \overset{i_X}{\hookrightarrow} X * Y$ null homotopic. ~~St~~ So does $Y \hookrightarrow X * Y$

$x \mapsto [x, 0, y]$

proof: we have factorization

$$X \to X * \{y_0\} \to X * Y.$$
$$\overset{\|}{CX}$$

Notation: $X * Y = \{(1-t)x \oplus ty \mid 0 \le t \le 1, (x,y) \in X \times Y\}/\sim$

~~when~~ $1 \cdot x \oplus 0y = 1 \cdot x \oplus 0y_1$ all $y, y_1 \in Y$

$0x \oplus 1 \cdot y = 0 \cdot x_1 \oplus 1 \cdot y$ all $x, x_1 \in X$.

For topological group G. Define

$$G^{*n} = \underbrace{G * G * \cdots * G}_{n}$$

$= \{t_1 g_1 \oplus t_2 g_2 \oplus \cdots \oplus t_n g_n \mid \sum t_i = 1, t_i \ge 0, g_i \in G\}/\sim$

$t_1 g_1 \oplus \cdots \oplus 0 g_i \oplus \cdots t_n g_n = t_1 g_1 \oplus \cdots \oplus 0 g_i' \oplus \cdots \oplus t_n g_n$

for all $g_i, g_i' \in G$.

G^{*n} has free right G-action

$(t_1 g_1 \oplus \cdots \oplus t_n g_n) \cdot g = t_1 g_1 g \oplus t_2 g_2 g \oplus \cdots \oplus t_n g_n g$

$EG = \operatorname{colim}_n G^{*n}$, with

$G^{*n} \hookrightarrow G^{*(n+1)}$

$(t_1 g_1 \oplus \cdots \oplus t_n g_n) \mapsto (t_1 g_1 \oplus \cdots \oplus t_n g_n \oplus 0 g_{n+1})$

EG is a free G-space which is weakly contractible: Any $f: S^n \to EG$

has factorization

$$\begin{tikzcd} S^n \arrow[rr, "f"] \arrow[dr, "\bar{f}"'] & & EG \\ & G^{*k} \arrow[ur] & \end{tikzcd}$$

for large k, and $\bar{f}$ is null homotopic since $G^{*k} \hookrightarrow G^{*(k+1)} = G^{*k} * G$ null homotopic.

$BG = EG/G$. Since EG is G-CW complex. (EG, p, BG) is a principal bundle.

BG is called classifying space.

prop. For a given G, BG is unique up to homotopy.

proof: Suppose $\xi = (EG, p, BG)$ $\xi' = (EG', p', BG')$ are universal bundle.

$$\begin{tikzcd} EG \arrow[r] \arrow[d] & EG' \arrow[r] \arrow[d] & EG \arrow[d] \\ BG \arrow[r, "g"] & BG' \arrow[r, "f"] & BG \end{tikzcd}$$

$\exists f, g.$ $f^* \xi = \xi'$, $g^*(\xi') = \xi$.

$\Rightarrow g^* f^*(\xi) = \xi.$ $\Rightarrow f \circ g \simeq 1$

Also $g \circ f \simeq 1$ $\Rightarrow BG \simeq BG'$.

Or prove by Yoneda lema

$$\begin{tikzcd} & {[-, BG]} \\ \mathrm{Prin}_G(-) \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & \\ & {[-, BG']} \end{tikzcd}$$

#

Prop. $p: E \to B$ universal, then $\Omega BG \simeq G$.

$\to \pi_n(EG) \to \pi_{n}(BG) \to \pi_{n-1}(G) \to \pi_{n-1}(EG) \to \cdots$

$\pi_n(EG) = 0$, $\pi_{n-1}(EG) = 0$

$\Rightarrow \pi_n(BG) = [S^n, BG] = [S^{n-1}, \Omega BG] = [S^{n-1}, G]$

$\Rightarrow \Omega BG \simeq G$ by whitehead thm.

#

Eg. (1) $[\Sigma X, BG] \cong [X, \Omega BG] \cong [X, G]$.

In particular

~~$[S^{n+1}, B]$~~ $\mathrm{Prin}_G(S^{n+1}) \cong [S^{n+1}, BG] \cong \pi_n(G)$

(2) When G discrete. $\pi_i(G) = 0$ $i \geq 0$

$\pi_0(G) = G$.

$BG \simeq K(G,1)$.

Eg. ① $S^1 = K(\mathbb{Z}, 1)$

Universal $\mathbb{Z}$-bundle $\mathbb{R} \xrightarrow{\exp} S^1$

② $\mathbb{Z}/n$ act on S^{2m-1} by

$S^{2m-1} \times \mathbb{Z}/n \to S^{2m-1}$

$(z_1, \cdots z_m), \rho_n \mapsto (z_1 \rho_n, z_2 \rho_n, \cdots, z_m \rho_n)$

$\rho_n = e^{\frac{2\pi i}{n}}$

$$\begin{array}{ccccccc} S^{2m-1} & \hookrightarrow & S^{2m+1} & \hookrightarrow & S^{2m+3} & \hookrightarrow \cdots & S^\infty \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L_n^{2m-1} & \hookrightarrow & L_n^{2m+1} & \hookrightarrow & L_n^{2m+3} & \hookrightarrow \cdots & L_n^\infty = B\mathbb{Z}/n \end{array}$$

universal $\mathbb{Z}/n$-bundle

When $n=2$, S^∞, $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$

③ Hopf fibration $S^1 \to S^{2m+1} \to \mathbb{C}P^m$

$$\begin{array}{ccccccc} S^{2m+1} & \hookrightarrow & S^{2m+3} & \hookrightarrow & \cdots & S^\infty & \\ \downarrow & & \downarrow & & & \downarrow & \\ \mathbb{C}P^m & \hookrightarrow & \mathbb{C}P^{m+1} & \hookrightarrow & \cdots & \mathbb{C}P^\infty & = BS^1 = BU(1) \end{array}$$

$\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$

1st chern class: $Vect^1_{\mathbb{C}}(X) = Prin_{U(1)}(X) = [X, BU(1)] = [X, \mathbb{C}P^\infty] = [X, K(\mathbb{Z},2)] = H^2(X, \mathbb{Z})$

1st S-W class: $Vect_{\mathbb{R}}(X) = Prin_{O(1)}(X) = [X, BO(1)] = [X, \mathbb{R}P^\infty] = [X, K(\mathbb{Z}/_2, 1)] = H^1(X, \mathbb{Z}/_2)$

~~Group cohomology (Discrete group)~~

Group cohomology (discrete)

A an abelian group, the (untwisted) group cohomology with coefficient A is defined as

$H^*(G; A) = H^*(BG; A)$ (singular cohomology)

well-defined since BG unique up to homotopy

If M a $\mathbb{Z}(G)$ module, a $\mathbb{Z}(G)$ resolution:

$0 \leftarrow M \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \leftarrow \cdots$

a long exact sequence of $\mathbb{Z}(G)$-mod.

Property: Given two resolution and a $\mathbb{Z}(G)$ map $f: M \to N$

$$\begin{array}{ccccccccc} 0 \leftarrow & M & \leftarrow & C_0 & \leftarrow & C_1 & \leftarrow & C_2 & \leftarrow \cdots \\ & f\downarrow & & \downarrow f_0 & & \downarrow f_1 & & \downarrow & \\ 0 \leftarrow & N & \leftarrow & D_0 & \leftarrow & D_1 & \leftarrow & D_2 & \leftarrow \cdots \end{array}$$

there is lifting $\{f_i\}$ of f, up to chain homotopy.

Group cohomology with twisted coefficient M is

$\mathrm{Ext}^i_{\mathbb{Z}G}(M, A) = H^i(C^*(M,A))$

with $C^n(M,A) = \mathrm{Hom}(C_n, A)$