Eric

Moduli space of flat connections in a principal bundle $\begin{array}{c} P \circlearrowleft G \\ \pi \downarrow \\ M \end{array}$

A connection A on $\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M \end{array}$ is a splitting map for

the exact seq $\quad 0 \to V \underset{A}{\rightleftarrows} TP \to \pi^* TM \to 0$

$T^{hor}P \subset TP$
$= \ker A$

require $\forall g \in G, p \in P \qquad A_p : T_pP \to V_p$

(*) $\quad (A_p(v))_g = A_{pg}(v \cdot g) \qquad$ G-equivariance

$\rightsquigarrow Ad_{g^{-1}}(A_p(v)) = A_{p \cdot g}(v \cdot g)$

Let $\mathfrak{g} = Lie(G)$

$P \times \mathfrak{g} \to V$

$(p, \xi) \mapsto \frac{\partial}{\partial t}\Big|_{t=0} (p \cdot \exp(t\xi))$

Taking $Hom_G(-, \underbrace{P \times \mathfrak{g}}_{\mathfrak{g}})$ we see

$0 \to Hom_G(\pi^* TM, \mathfrak{g}) \to Hom_G(TP, \mathfrak{g}) \to Hom_G(\mathfrak{g}, \mathfrak{g}) \to 0$

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\in}{A}$

$\qquad\qquad\qquad\qquad\qquad id_{\mathfrak{g}} + Hom_G(\pi^* TM, \mathfrak{g})$

$\pi^* TM \to TM$

So $\exists$ iso $\quad Hom_G(\pi^* TM, \mathfrak{g}) \cong \Gamma(TM, \underbrace{TM \times_M P}_{\pi^* TM} \times_G \mathfrak{g})$

$= \Gamma\, Mor_{V.B/M}(TM, P \times_G \mathfrak{g}) \otimes$

$= \underbrace{\Omega^1(M, Ad_P)}_{\Omega^1(M, \mathfrak{g}) \text{ (by abuse of notations)}} := \Omega^1(M) \otimes \Gamma(M, P \otimes_G \mathfrak{g})$

A connection $\quad d_A : \Omega^k(M, \mathfrak{g}) \to \Omega^{k+1}(M, \mathfrak{g})$

$$\alpha \longmapsto d\alpha + \underbrace{[A \wedge \alpha]}_{\text{wedge of form part} \otimes [\,,\,] \text{ of Lie part}}$$

$A = s \otimes \xi$
$\alpha = s' \otimes \xi'$ $\quad \Rightarrow [A \wedge \alpha] = (s \wedge s') \otimes [\xi, \xi']$

$$d_A d_A \alpha = d_A(d\alpha + [A \wedge \alpha])$$

$$= \cancel{dd\alpha} + \underbrace{d[A \wedge \alpha] + [A \wedge d\alpha]}_{[dA \wedge \alpha]} + \underbrace{[A \wedge [A \wedge \alpha]]}_{\frac{1}{2}[[A \wedge A] \wedge \alpha] \text{ by Jacobi}}$$

$$= [(dA + \tfrac{1}{2}[A \wedge A]) \wedge \alpha]$$

for A, $\quad F_A = dA + \frac{1}{2}[A \wedge A]$ - curvature

A is flat if $F_A = 0$ $\quad$ ($\overset{\text{by Frobenius thm}}{\Longleftrightarrow}$ $T^{hor} \subset TP$ is an integrable distribution)

def A map between principal bundles $P \xrightarrow{\varphi} P'$ is a G-equivariant bundle map

a gauge transform is a bundle isom $P \xrightarrow{\varphi} P$

$\varphi(p) = p \cdot g \Rightarrow \varphi(p \cdot h) = \varphi(p) \cdot h = p \cdot gh = ph(h^{-1}gh)$

$\varphi : P \to P$

$$\begin{tikzcd}
0 \arrow[r] & P \times \mathfrak{g} \arrow[r] \arrow[d, "\varphi_*"] & TP \arrow[r] \arrow[d, "\varphi_*"] & \pi^* TM \arrow[r] \arrow[d, "\varphi \times id_{TM}"] & 0 \\
& P \times \mathfrak{g} \arrow[r] & TP \arrow[r] \arrow[l, bend left, "A"] & \pi^* TM \arrow[r] & 0
\end{tikzcd}$$

$\varphi : P \to G$ with action by conjugation

$\tilde{\varphi} : P \to P$

$p \cdot g \to pg^{-1} \varphi(pg^{-1}) g$

if A, B are 2 connections on P, $A \sim B$ if $\exists\, \varphi : P \to P$ s.t. $\varphi^* A = B$

Note: if $F_A = 0$ then $F_{\varphi^* A} = 0$

$\varphi^* A = TP \to P \times \mathfrak{g}$

$v \longmapsto \varphi^{-1}_* A(\varphi_* v)$

<u>Ex:</u> $G = \mathbb{R}$, $\mathcal{M}^{triv}(M, G)$ ← only connections in the triv. G-bundle

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & M \times G \arrow[d] \\
M \arrow[r] & M
\end{tikzcd}$$

$\Omega^1(M,\mathbb{R}) = \Omega^1(M)$ $\quad F_A = dA \quad [A\wedge A]=0$

$$\mathcal{M}^{triv}(M,\mathbb{R}) = \{A\in\Omega^1(M) \mid dA=0\} \big/ A\sim A+d\alpha$$

$$\varphi_*^{-1}\hat{A}\varphi_* + \varphi_*^{-1} d\varphi_*$$

Holonomy in P with connection A

recall $\gamma:[0,1]\to M$ and $p\in P_{\gamma(0)}$

$Pt_\gamma : P_{\gamma(0)} \longrightarrow P_{\gamma(1)=\gamma(0)}$

A - principal connection $\qquad Pt_\gamma(p) = p\cdot g$

$Pt_\gamma(p\cdot h) = pt(p)\cdot h = p\cdot gh = ph(h^{-1}gh)$

$Hol_p(A,\gamma) = g \qquad Hol(A,\gamma)\subseteq G$

$\varphi : P\to P \qquad Hol_{\varphi(p)}$

Construction of $\mathcal{M}(M,G)$

$\mathcal{M}(M,G) := \{(P,A)$ - principal G-bundles over M with a flat connection $A\} \big/ \sim$

gauge transf.

$\Phi : \mathcal{M}(M,G) \longrightarrow Hom(\pi_1(M),G)\big/G$ acts by conjugation

$[P,A] \longmapsto ([\gamma] \longmapsto Hol_p(A,\gamma)^{-1})$

Claim: this is a bijection!

need to check : gauge equiv. classes

Talk 2

$$\mathcal{M}(M,G) = \overset{\text{set}}{\{}\text{principal bundles } P \text{ with flat conn. } A\}\Big/\text{gauge equivalence}$$

Thm $\mathcal{M}(M,G) \xrightarrow{\Psi} \mathrm{Hom}(\pi_1 M, G)/G$ acts via conjugation

$$[P,A] \longmapsto \left([\gamma] \mapsto \mathrm{Hol}_p(A,\gamma)^{-1}\right)$$

to get a group hom.

- $p \in P$
- $\mathrm{Hol}_p(A,\gamma)$ only depends on $[\gamma]$
- bijective

Ex: G abelian:

then conj. trivial

$$\mathcal{M}(M,G) \cong \mathrm{Hom}(\pi_1 M, G) \cong \mathrm{Hom}(H_1(M), G) \cong H^1(M,G)$$

if $G = \mathbb{R}$

G discrete $\begin{array}{c} P \\ \downarrow \\ M \end{array}$

$$\mathcal{M}(M,G) \cong [M, BG] \cong \mathrm{Hom}(\pi_1 M, G)/G$$

$G = U(1)$ or $\mathbb{C}^*$

$$\mathcal{M}(M,G) = \{\text{flat bundles}\}/\sim \; = H^1(M, U(1)) \leftarrow \text{torsion part of } H^2(M,\mathbb{Z})$$

$\cap$; exp. from LES in $H^*(\;, \mathbb{Z} \to \mathbb{R} \to S^1)$

$$\{\text{all bundles}\}/\sim \; = H^2(M, \mathbb{Z})$$

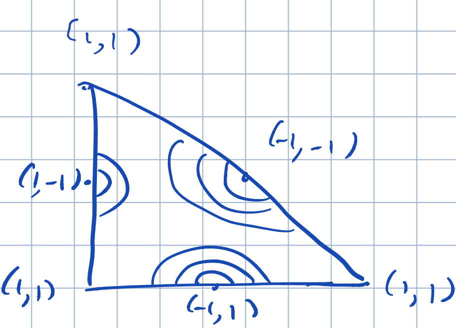
$\mathbb{R}P^3$ $\quad \mathcal{M}(\mathbb{R}P^3, U(1)) = * \quad *$

pt, S^1 $\quad \mathcal{M}(\text{pt or } S^1, U(1)) = *$

$G = SU(2)$ $\quad \mathcal{M}(S^1, SU(2)) =$ 

$G = SU(2)$, M = torus

$$\mathcal{M}(M, SU(2)) = \{a, b \in SU(2) \mid ab = ba\}$$



M, G fixed

T a cell decomp

$$\mathcal{M}(T,G) = \left. \left\{ U : \{1\text{-cells}\} \to G \;\middle|\; \forall \;\; U(e_1)\cdots U(e_n) = 1 \right\} \right/ U \sim U' \text{ if } \exists\, g : \{0\text{-cells}\} \to G$$

s.t. for each edge e $\quad u \xrightarrow{e} v$,

$$U(e) = g(v)\, U'(e)\, g(u)^{-1}$$

$$\begin{array}{cc} u & P \\ \downarrow & \downarrow \\ u \cdot g(u) & P \end{array}$$

<u>Symplectic structure</u> on $\mathcal{M}(\Sigma, G)$

Atiyah-Bott 2-form.

$M = \Sigma$ compact oriented surface without boundary

G - Lie group with bi-invariant metric $\langle\,,\,\rangle$

Fix an iso class $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ $\quad \mathcal{A} = (P, A)$ connections $= \Omega^1(M, \mathrm{Ad}_P)$

$A \in \mathcal{A}$, $\quad T_A \mathcal{A} = \Omega^1(M, \mathrm{Ad}_P)$

AB 2-form $\qquad \omega_{AB}(\alpha, \beta) = \int_\Sigma \langle \alpha \,\hat{,}\, \beta \rangle \quad$ where $\alpha, \beta \in T_A \mathcal{A}$

$$\mathcal{G}(P) = \mathrm{Aut}(P)$$
$$\mathrm{Lie}(\mathcal{G}(P)) = \Omega^0(M, \mathrm{Ad}_P)$$

$$\begin{aligned} \Omega^2(M, \mathrm{Ad}_P) \times \Omega^0(M, \mathrm{Ad}_P) &\longrightarrow \mathbb{R} \\ (F, h) &\longmapsto \int_\Sigma \langle F \,\hat{,}\, h \rangle \end{aligned}$$

$$\mathrm{Lie}(\mathcal{G}(P))^* \cong \Omega^2(M, \mathrm{Ad}_P)$$

$$\begin{aligned} \mathcal{A} &\xrightarrow{\mu} \mathrm{Lie}(\mathcal{G}(P))^* \\ (P, A) &\longmapsto -F_A \end{aligned}$$

Claim: $\mathcal{A} \circlearrowleft \mathcal{G}(P)$ is Hamiltonian with moment map μ.

$\mu^{-1}(0) =$ flat connections in P.

symp reduction $=$ moduli space $\mathcal{M}(\Sigma, G)$

$\rightsquigarrow$ $\underline{\omega}_{AB}$ - symp. structure

<u>Thm</u> For X compact Kähler, G Lie group admitting bi-invariant metric

then $\mathrm{Hom}(\pi_1 X, G)/G$ has a symp. structure

Yael Karshon

An alg. proof of

symp str on moduli space