Recall:

Applications

· splitting principle. for $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ a v.bdle,

$$
\begin{tikzcd}
p^*E \cong L_1 \oplus \cdots \oplus L_n \arrow[r] \arrow[d] & E \arrow[d] \\
Y \arrow[r, "p"'] & X
\end{tikzcd}
$$

p^* injective on cohomology.

$c_1(L_i) = x_i$

$1 + c_1(L_i)$

$$c(p^*E) = \prod_i c(L_i)$$

$$p_j = (-1)^j c_{2j}$$

$$\int \hat{A}(TM)\, ch(S/\Delta)$$

NEXT TIME

- Hirzebruch signature thm
- Hirzebruch-Riemann-Roch

Richard talk II

Hirzebruch signature thm & Hirzebruch-Riemann-Roch

$g(M_1 \times M_2) = g(M_1)\, g(M_2)$ ← multiplicative genus

$c(E) = 1 + c_1(E) + c_2(E) + \dots$

$c(E_1 \oplus E_2) = c(E_1)\, c(E_2)$

Splitting principle

$$\begin{tikzcd} L_1 \oplus \dots \oplus L_k \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r, "p"'] & X \end{tikzcd}$$

p^* injective on cohom.

sketch of proof.

$$\begin{tikzcd} L_E \arrow[r, hook] \arrow[dr] & p^*E \arrow[r] \arrow[d] & E \arrow[d] \\ & \mathbb{P}(E) \arrow[r, "p"'] & X \end{tikzcd}$$

$L_E = \left\{ (\ell, v) \in \mathbb{P}(E) \times E \;\middle|\; v \in \ell \right\}$

$L_E \oplus E' = p^*E$ — split off one line bdle!

then iterate

$\boxed{\;}$

E "=" $L_1 \oplus \dots \oplus L_k$

$c(E) = c(L_1) \cdots c(L_k) = \prod_{i=1}^{k} (1 + x_i)$ ⊜

$c(L_i) = 1 + \underbrace{c_1(L_i)}_{=: x_i} = 1 + x_i$

⊜ $\underset{\uparrow \atop c_0}{1} + \underbrace{\sum_i x_i}_{c_1} + \underbrace{\sum_{i \neq j} x_i x_j}_{c_2} + \underbrace{\sum_{\text{distinct}} x_{i_1} x_{i_2} x_{i_3}}_{c_3} + \dots$

Idea: mult. genera are products $\prod_{i=1}^{k} f(x_i)$ where E "=" $L_1 \oplus \dots \oplus L_k$
where f power series

Example: $c \rightsquigarrow f(x) = 1 + x$

$G_1 \longleftrightarrow f_1$

$G_2 \longleftrightarrow f_2$

$G_1 \wedge G_2 \longleftrightarrow f_1 \cdot f_2$

$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C})$

"even power series" $\rightsquigarrow$ power series for p_j (instead of c_j)

$\hat{A}$ genus $\longleftrightarrow \dfrac{\frac{\sqrt{z}}{2}}{\sinh \frac{\sqrt{z}}{2}}$

L -genus $\longleftrightarrow \dfrac{\sqrt{z}}{\tanh \sqrt{z}}$

Hirzebruch signature thm

Let M be a $4m$ - mfd, oriented, cpt

$$\boxed{\mathrm{sign}(M) = \int_M L(TM)}.$$

A-S: $\quad \mathrm{Ind}(D) = \int_M \hat{A}(TM)\, ch(V/S)$

$ch(S) = \sum_{i=1}^{k} e^{c_1(L_i)}$

Proof sketch

$V = \Omega^* TM = \Lambda^*(T^*M) \otimes \mathbb{C}$ it is a Clifford bundle

$\cong Cl(TM) \otimes \mathbb{C}$

$\cong \mathrm{Hom}(S,S) \cong S \otimes S$ spin bundle

$S(TM)$ - spin repn of $Cl(TM)$

$\mathrm{Hom}(S,S)$

$ch(V/S) \qquad V = S \otimes ? \qquad ch(?)$

$V = S \otimes S \qquad ch(S)$

Claim: $ch(S) = 4^m \mathcal{G}(TM)$

$\mathcal{G}(TM) \longleftrightarrow \cosh \frac{\sqrt{z}}{2}$

Idea $ch(S(\bigoplus_i V_i)) = ch(\bigotimes_i S(V_i)) = \prod_i ch(S(V_i))$

$ch(S(V))$ $\quad V$ rk 2

vaguely: $S = S^+ \oplus S^-$, $S^- = S^{+*}$

$ch(S^+) = e^{c_1}$

$ch(S^+ \oplus S^-) = e^{c_1} + e^{-c_1} = 2\cosh c_1$

$2^{2m} = 4^m$

$\hat{A} \longleftrightarrow \frac{\sqrt{z}/2}{\sinh \sqrt{z}/2}$ $\quad ch(S) = 4^m \cosh \frac{\sqrt{z}}{2}$

$\to \hat{A} \wedge ch(S) \longleftrightarrow \frac{\sqrt{z}/2}{\tanh \sqrt{z}/2} \cdot 4^m = \sum a_j \frac{z^j}{4^j}$ | in product $(z_1, \ldots z_m)$ get terms like $\prod_i \left(\frac{z_i}{4}\right)^{j_i}$, $\sum j_i = 4m$

$\int_M [\hat{A} \wedge ch(S)]_{4m}$ $\quad \ldots = \frac{1}{4^m} \prod z_i^{j_i}$

therefore $\frac{\sqrt{z}/2}{\tanh \sqrt{z}/2} \longleftrightarrow \frac{1}{4^m} \frac{\sqrt{z}}{\tanh \sqrt{z}}$ on $4m$-degree piece

So: $[\hat{A} \wedge ch(S)]_{4m} \longleftrightarrow \left[\frac{\sqrt{z}}{\tanh \sqrt{z}}\right]_{4m} = [L]_{4m}$

So: $\int_M \hat{A}(TM) \wedge ch(S) = \int_M L(TM)$ $\quad$ RHS ✓

LHS: want $\mathrm{sign}(M) = \mathrm{Ind}(D)$.

Aside: Hodge-$*$

M^n - Riemannian

flat case: $* : \Omega^k \to \Omega^{n-k}$

$dx_1 \wedge dx_2 \to dx_3 \wedge \dots \wedge dx_n$

generally: $\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta$ $\qquad *^2 = (-1)^{k(n-k)} Id$

Hermitian, positive

$d^* = - * d *$ - adjoint of d

$D = d + d^*$

$D^2 = \cancel{d^2} + d d^* + d^* d + \cancel{(d^*)^2} = \Delta$ ← Laplacian

Back to M^{4m}

$\ker D \subseteq \ker D^2 = \ker \Delta$

$\quad\quad\quad\quad\quad\quad\quad\quad \| \;$ Hodge - de Rham

$\quad\quad\quad\quad\quad \ker d \cap \ker d^* \subseteq \ker D$

$\Rightarrow \ker D = \ker \Delta$

$\quad\quad\quad\quad\quad\quad \|$

$\quad\quad\quad\quad\quad\quad \mathcal{H}$ - harmonic forms

Grading

$\varepsilon = i^{2n + k(k-1)} * \qquad \varepsilon^2 = Id$

So, $\Omega^* = \cancel{\Omega^* \otimes \mathbb{C}}$ decomposes as ± 1 eigenspaces of ε

Also, can show: D swaps the grading and Δ preserves grading

$\Delta = \Delta^+ + \Delta^-$

$\ker \Delta = \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

$\mathrm{Ind}(D) = \dim \ker D^+ - \dim \ker D^-$

$= \dim \mathcal{H}_+ - \dim \mathcal{H}_-$

Consider $\Omega^k \oplus \Omega^{4m-k} \circlearrowleft \varepsilon$; let $k < 2m$

If α^{ω} and $\alpha \in (\Omega^*)^+$ then $\alpha = \beta + \varepsilon(\beta)$; then $\beta - \varepsilon(\beta) \in (\Omega^*)^-$

So, can go from $(\Omega^*)^+$ to $(\Omega^*)^-$ in an isomorphic way when $k < 2n$.

$\beta + \varepsilon(\beta) \longrightarrow \beta - \varepsilon(\beta)$

So: $\dim \mathcal{H}^k_+ + \dim \mathcal{H}^{4m-k}_+ = \dim \mathcal{H}^k_- + \dim \mathcal{H}^{4m-k}_-$

- Done except $k = 2m$

$\Rightarrow \mathrm{Ind}(D) = \dim \mathcal{H}^{2m}_+ - \dim \mathcal{H}^{2m}_-$ $=$

$\langle \alpha, \alpha \rangle = \int_M \alpha \wedge * \alpha = \pm \int_M \alpha \wedge \alpha$ if $\alpha \in \mathcal{H}^{2m}_\pm$

$\varepsilon = *$ on $2m$-forms

$=$ $\mathrm{sign}(M)$ $\square$.

Richard talk IV

$x_1 \cdots x_n$ ← Chern roots of E

$$\begin{tikzcd} L_1 \oplus \cdots \oplus L_n \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

power series $Q(x)$

def: $Q(x_i)$

Hirzebruch signature thm

$$\operatorname{sign}(M) = \int_M L(TM)$$

L assoc to $\dfrac{\sqrt{z}}{\tanh\sqrt{z}}$

signature of the intersection form

Hirzebruch-Riemann-Roch

Riemann-Roch Let C be a $\mathbb{C}$-curve and $\mathcal{L} = \mathcal{L}(D)$ a line bundle then

line bundle   divisor

$$h^0(\mathcal{L}) - h^1(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g$$

$$[\ h^0(D) - h^0(K-D) = \deg D + 1 - g \quad \text{- Serre duality}\ ]$$

genus
$$g = h^1(\mathcal{O}) = (\chi_{hol}(\mathcal{O}) - 1)\cdot(-1)^{\dim}$$

$$h^j = \dim H^j_{\bar\partial}(-) = h^{0,j}(W)$$

Ex: $\underbrace{h^0(\mathcal{O})}_{1} - \underbrace{h^0(K)}_{g} = 0 + 1 - g$

Hirzebruch-Riemann-Roch

W - any holom. v.b. on M^n ($\mathbb{C}$-mfd)

Then $$\chi_{hol}(W) = \int_M td(TM) \wedge ch(W)$$

here $\chi_{hol}(W) = \sum (-1)^j h^j(W)$ - holomorphic Euler char.

$\mathbb{C}\otimes TM = T^{1,0}M \oplus T^{0,1}M$

$V = \Omega^{0,*}M$

$$L = \mathrm{Hom}_{\mathbb{C}\ell}(V, \bar{V})$$
$$\cong \Omega^{0,n}(M)$$
$$= \Lambda^n (T^{0,1}M)^*$$

$$c_1(L) = c_1((T^{0,1}M)^*)$$
$$= c_1(T^{1,0}M)$$

$$ch(V/S) \overset{\dots}{=} \sqrt{ch(L/S)} = e^{c_1(L)/2}$$

$$\hat{A} \to \frac{z/2}{\sinh z/2} \qquad \hat{A}\wedge ch(V/S) \to \frac{z/2}{\sinh z/2}\cdot e^{z/2} = \frac{z}{1-e^{-z}}$$

Chern genus, not Pontrjagin genus $\qquad \rightsquigarrow td(TM)$

$$D = \sqrt{2}(\bar\partial + \bar\partial^*)$$

$$D^2 = 2(\underset{\substack{\| \\ 0}}{\bar\partial^2} + \underbrace{\bar\partial\bar\partial^* + \bar\partial^*\bar\partial}_{} + \underset{\substack{\| \\ 0}}{\bar\partial^{*2}})$$

$= 2\Delta_{\bar\partial} \quad = \Delta$

"holom. Laplacian"

actually need D + order 0
$\}$ homotopy
$\mathcal{D}$

$$\ker D = \ker \Delta_{\bar\partial}$$

$$\mathrm{ind}\, D = \cancel{\chi}(\Omega^{0,*}\otimes W, \bar\partial) = \chi_{hol}(W)$$
$$= h^{0,0}(W) - h^{0,1}(W) + h^{0,2}(W) ----$$

Atiyah-Singer:

$$\chi_{hol}(W) = \int_M \hat{A}(TM)\wedge ch(V\otimes W/S)$$
$$= \int_M \hat{A}\wedge ch(V/S)\wedge ch(W) = \int_M td(TM)\wedge ch(W) \qquad \square$$

Ex: R-R for curves

$W = \mathcal{L}(D)$ $\qquad td = 1 + \frac{c_1(TX)}{2}$

$ch(\mathcal{L}) = 1 + c_1(\mathcal{L})$

$$\chi_{hol}(\mathcal{L}) = h^0(D) - h^1(D)$$
$$= \int_M \frac{c_1(TX)}{2} + c_1(\mathcal{L})$$
$$= \int_M \tfrac{1}{2} c_1(-K) + c_1(D) = \tfrac{1}{2}\deg(-K) + \deg(D)$$
$$= \tfrac{1}{2}(2-2g) + \deg(D)$$
$$= \deg(D) + 1 - g$$

R-R for surfaces

$W = \mathcal{L}(D)$

$$td(TX) = 1 + \tfrac{1}{2} c_1(TX) + \tfrac{1}{12}\left(c_1(TX)^2 + c_2(TX)\right)$$
$$ch(\mathcal{L}) = 1 + c_1(\mathcal{L}) + \tfrac{1}{2} c_1(\mathcal{L})^2$$

$$\chi_{hol}(D) = h^0(D) - h^1(D) + h^2(D)$$
$$= \int_X \tfrac{1}{2} c_1(D)^2 + \tfrac{1}{2} c_1(D)\, c_1(TX) + \tfrac{1}{12}\left(c_1(TX)^2 + c_2(TX)\right)$$
$$= \tfrac{1}{2}(D\cdot D - D\cdot K) + \tfrac{1}{12}(K\cdot K + e(X))$$

$c_1(TX) = -c_1(K)$ $\qquad K = \Lambda^2 T^*M$

$$D = 0 \rightsquigarrow \chi(0) = \tfrac{1}{12}(K\cdot K + e(X))$$

$$\chi(D) = \tfrac{1}{2} D\cdot(D-K) + \underset{1+g}{\underset{\shortparallel}{\chi(0)}}$$