LAST TIME

- X - vector field on M

$\rightsquigarrow X : C^\infty(M) \to C^\infty(M)$ s.t. $X(fg) = f\,X(g) + g\,X(f)$ } - derivation

Lie bracket $[X,Y] = X \circ Y - Y \circ X$ - new vector field

(as operators on $C^\infty(M)$)

A vector bundle $\begin{array}{c} E \\ \downarrow p \\ M \end{array}$ is the data of
- a manifold E ("total space")
- a manifold M ("base")
- projection $p : E \to M$ (smooth surjective map)

such that
- $\forall x \in M$, $p^{-1}(x)$ is a m-dim. vector space
- $\forall x \in M$ there is a nbhd U and a diffeo $\varphi_U : p^{-1}(U) \xrightarrow{\simeq} U \times \mathbb{R}^m$ ($p^{-1}(U) \subset E$)

s.t. φ_U maps $p^{-1}(y)$ into $\{y\} \times \mathbb{R}^m$ $\forall y \in U$ as a linear isomorphism.

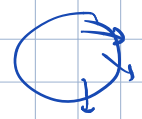
} local triviality condition

< loc. trivialization (U, φ_U) near x is only required to exist; it is not fixed/a part of data of the bundle >

def The Lie bracket of two vector fields X, Y is the vector field $[X, Y]$. ①

Ex: $M = \mathbb{R}$, $X = f\frac{d}{dx}$, $Y = g\frac{d}{dx}$ $\Rightarrow [X,Y] = (fg' - gf')\frac{d}{dx}$

One-parameter groups of diffeomorphisms

intuition: wind velocity vector field on S^2  moves a particle at x to a new point $\varphi_t(x)$ after time t. after time s, it is at $\varphi_s(\varphi_t(x)) = \varphi_{s+t}(x)$

def A one-parameter group of diffeomorphisms ("flows") of a manifold M is a smooth map $\varphi : \mathbb{R} \times M \to M$, $(t, x) \mapsto \varphi_t(x)$ s.t.

- $\varphi_t : M \to M$ is a diffeomorphism
- $\varphi_0 = id$
- $\varphi_{s+t} = \varphi_s \circ \varphi_t$

idea: a vector field gives rise to a 1-param grp of diffeo, under certain assumptions.

- Given a 1-param grp of diffeo φ_t, we can produce a vector field out of it:

for $f \in C^\infty(M)$, set $X_a(f) := \frac{d}{dt}\Big|_{t=0} f(\varphi_t(a))$

- this X_a satisfies Leibnitz property: $X_a(fg) = \frac{d}{dt}\Big|_{t=0} f(\varphi_t(a))\, g(\varphi_t(a))$

use $\varphi_0(a) = a$ $\quad = f(a)\frac{d}{dt}\Big|_{t=0} g(\varphi_t(a)) + g(a)\frac{d}{dt}\Big|_{t=0} f(\varphi_t(a))$

$= f(a)\, X_a(g) + g(a)\, X_a(f)$

$\Rightarrow X_a$ is a tangent vector at a.

- Locally: $\varphi_t(x_1, \dots, x_n) = (y_1(t,x), \dots, y_n(t,x))$

$$\frac{d}{dt}\Big|_{t=0} f(y_1, \dots, y_n) = \sum_i \frac{\partial f}{\partial y_i}(y) \underbrace{\frac{dy_i}{dt}(x)\Big|_{t=0}}_{c_i(x)} = \sum_i c_i(x)\frac{\partial f}{\partial x_i}(x) = X(f)$$

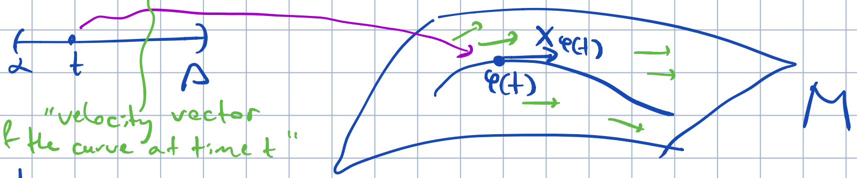
with $X = \sum_i c_i(x)\frac{\partial}{\partial x_i}$ – vector field.

- We want to reverse this process and go from a vector field to the diffeomorphism.
  - First, we want to track the "trajectory of a single particle".

<u>def</u> An <u>integral curve</u> of a vector field X is a smooth map

$\varphi : (\alpha, \beta) \to M$ such that $\boxed{\underbrace{D\varphi_t\left(\frac{d}{dt}\right)}_{\text{"velocity vector of the curve at time } t\text{"}} = X_{\varphi(t)}}$ $(*)$

$\cap$
$\mathbb{R}$



<u>Example</u>[#] $M = \mathbb{R}^2$ with coordinates (x, y)

and $X = \frac{\partial}{\partial x}$. The derivative $D\varphi$ of a smooth fun. $\varphi(t) = (x(t), y(t))$

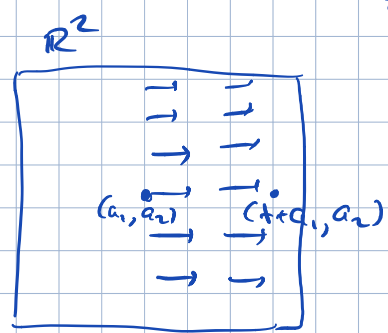
is $D\varphi_t\left(\frac{d}{dt}\right) = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y}$. So, the equation $(*)$ is:

$$\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = 0 \end{cases} \Rightarrow \varphi(t) = (t + a_1, a_2)$$

system of ODE

a solution (general)

- particle at (a_1, a_2) is trasported in time t to $(t + a_1, a_2)$



<u>Theorem</u>[*] Given a vector field X on a manifold M and $a \in M$, there exists a unique <u>maximal</u> integral curve of X with $\varphi(0) = a$.

i.e. the interval (α, β) is maximal

For the proof, we need

10.4 in Hitchin

Thm (Picard-Lindelöf): Let $f(t, x)$ be a continuous function with values in $\mathbb{R}^n$ on $|t - t_0| \leq a$, $\|x - x_0\| \leq b$ and suppose f satisfies Lipschitz condition $\|f(t, x_1) - f(t, x_2)\| \leq L\, \|x_1 - x_2\|$. (for some L "Lipschitz constant")

If $\lambda = \sup |f(t, x)|$ and $h = \min(a, b/\lambda)$, then the differential equation $\frac{dx}{dt} = f(t, x)$, $x(t_0) = x_0$ has a unique solution for $|t - t_0| \leq h$.

Proof of THM* Consider (U_γ, ψ_γ) - chart around a. Then if $X = \sum_i c_i(x) \frac{\partial}{\partial x_i}$, the eq. $D\varphi_t\left(\frac{d}{dt}\right) = X_{\varphi(t)}$ can be written as a sys. of ODEs

$\frac{dx_i}{dt} = c_i(x_1, \dots, x_n)$. By Picard-Lindelöf, $\exists!$ sol. on some interval with init. cond. $(x_1(0), \dots, x_n(0)) = \psi_\gamma(a)$.

Suppose $\varphi, \varphi' : (\alpha, \beta) \to M$ any two integral curves with $\varphi(0) = \varphi'(0) = a$. $\forall x \in (\alpha, \beta)$, interval $[0, \cancel{x}]$ is compact $\Rightarrow$ can be covered by a fin. number of coord. charts, in each of which we can apply P-L to intervals $[0, \alpha_1], [\alpha_1, \alpha_2], \dots, [\alpha_n, x]$.

Uniqueness $\Rightarrow \varphi = \varphi'$ on $[0, \alpha_1]$ $\rightsquigarrow$ on $[\alpha_1, \alpha_2] \to \dots \rightsquigarrow$ on $[0, x]$ $\Rightarrow \varphi = \varphi'$ everywhere

$\rightsquigarrow$ then we take the maximal interval on which we can define φ. $\square$

- To find the 1-param. group of diffeo, we now let $a \in M$ vary. In Ex# above, the integral curve through (a_1, a_2) was $t \mapsto (t + a_1, a_2)$. This defines the group of diffeo $\varphi_t(x_1, x_2) = (t + x_1, x_2)$.

Theorem@ Let X be a vector field on a mfd M and for $(t, x) \in \mathbb{R} \times M$, let $\varphi(t, x) = \varphi_t(x)$ be the maximal integral curve of X through x. Then

1. the map $(t, x) \longmapsto \varphi_t(x)$ is smooth.
2. $\varphi_t \circ \varphi_s = \varphi_{t+s}$ wherever the maps are defined.
3. if M is compact, then $\varphi_t(x)$ is defined on the entire $\mathbb{R} \times M$ and gives a one-parameter group of diffeomorphisms.

We need the following result on smooth dependence of solutions on the initial conditions:

THM** (10.7 in Hitchin) If $f : [t_0 - a, t_0 + a] \times B(x_0, b) \to \mathbb{R}^n$ is C^k, $k \geq 1$, and $\frac{d}{dt}\alpha(t, x) = f(t, \alpha(t, x))$, $\alpha(t_0, x) = x$, then α is also C^k.

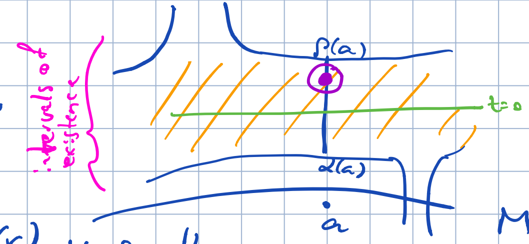
defined in an open nbhd of $\{t_0\} \times \overset{\text{open}}{B}(x_0, b)$

(case $k = \infty$: $f \in C^\infty \Rightarrow \alpha$ depends smoothly on init. cond.)

Proof of THM@ By THM*, $\forall a \in M$ we have an interval $(\alpha(a), \beta(a))$ on which maximal integral curve is defined. P-L THM (local existence) + continuous dependence on init. cond. x_0 (THM 10.5 in Hitchin) also implies that there is a solution for init. conditions in a nbhd of a.

So, the set

$V = \{(t,x) \in \mathbb{R}\times M : t \in (\alpha(x), \beta(x))\}$ is open.



- This the set on which $\varphi_t(x)$ is maximally defined

(1) Smooth dependence thm** says that $(t,x) \longmapsto \varphi_t(x)$ is smooth.

(2) Consider $\varphi_t \circ \varphi_s(x)$. fix s, vary t $\rightsquigarrow$ this is the unique int. curve of X through $\varphi_s(x)$

but $\varphi_{t+s}(x)$ is also an integral curve passing through $\varphi_s(x)$ at $t=0$ $\Longrightarrow$ $\varphi_t \circ \varphi_s(x) = \varphi_{t+s}(x)$

Also: $\varphi_t \circ \varphi_{-t} = Id$ $\Rightarrow$ we have a diffeo wherever it is defined.

(3) Consider the case when M is compact.

possibly non-max. interval

$\forall x \in M$ we have an open interval $(\tilde\alpha(x), \tilde\beta(x))$ containing 0 and an open nbhd $U_x \subset M$ s.t. $\varphi_t(y)$ is defined on $(\tilde\alpha(x), \tilde\beta(x)) \times U_x$. Cover M by $\{U_x\}_{x\in M}$ and take a finite subcovering $U_{x_1}, \dots, U_{x_N}$; set $I = \bigcap_{i=1}^{N} (\tilde\alpha(x_i), \tilde\beta(x_i))$ – open interval containing 0.

Thus, for $t \in I$, we have $\varphi_t : I \times M \to M$, defining (possibly non-maximal) int. curve through each $x \in M$, with $\varphi_0(x) = x$.

We need to extend to all real values t.

- define $\boxed{\varphi_t = (\varphi_{t/n})^n}$ multiplication = composition

  where n is large enough, so that $\frac{t}{n} \in I$.

  - this is well-defined, if we choose m s.t. $\frac{t}{m} \in I$, then

$$(\varphi_{t/m})^m = \left(\left(\varphi_{\frac{t}{m\cdot n}}\right)^n\right)^m = \left(\left(\varphi_{\frac{t}{m\cdot n}}\right)^m\right)^n = (\varphi_{t/n})^n$$

$$= \left(\varphi_{\frac{t}{m\cdot n}}\right)^{n\cdot m} \quad //$$

  - compatible with composition:

$$\varphi_t \circ \varphi_s = \left(\varphi_{\frac{t}{n}}\right)^n \left(\varphi_{\frac{s}{n}}\right)^n = \left(\varphi_{\frac{t+s}{n}}\right)^n = \varphi_{t+s}$$

choose n s.t. $\frac{t+s}{n}, \frac{t}{n}, \frac{s}{n} \in I$

$$\varphi_{t/n} \circ \varphi_{s/n} = \varphi_{\frac{t+s}{n}} = \varphi_{s/n} \circ \varphi_{t/n}$$ – commute!

□