LAST TIME: • flow $\varphi: \mathbb{R}\times M \to M$, $(t,x)\mapsto \varphi_t(x)$ s.t. $\varphi_{t+s} = \varphi_t \circ \varphi_s$, $\varphi_0 = id_M$, $\varphi_t : M \to M$ - diffeo,

• flow $\varphi_t \rightsquigarrow$ vector field X; $X(f) = \frac{d}{dt}\Big|_{t=0} \varphi_t^* f$

• integral curve of a vector field X - map $\varphi: (\alpha,\beta) \to M$ s.t. $D\varphi_t\left(\frac{d}{dt}\right) = X_{\varphi(t)}$
$(\alpha,\beta) \subseteq \mathbb{R}$

THM: given a vect. field X on M and a point $a\in M$, $\exists!$ a maximal int. curve $\varphi: (\alpha,\beta)\to M$ of X with $\varphi(0) = a$.

Theorem@ Let X be a vector field on a mfd M and for $(t,x)\in \mathbb{R}\times M$, let $\varphi(t,x) = \varphi_t(x)$ be the maximal integral curve of X through x. Then

1. the map $(t,x) \longmapsto \varphi_t(x)$ is smooth.
2. $\varphi_t \circ \varphi_s = \varphi_{t+s}$ wherever the maps are defined.
3. if M is compact, then $\varphi_t(x)$ is defined on the entire $\mathbb{R}\times M$ and gives a one-parameter group of diffeomorphisms.

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we need the following result on smooth dependence of solutions on the initial conditions:

THM** (10.7 in Hitchin) If $f: [t_0 - a, t_0 + a] \times B(x_0, b) \to \mathbb{R}^n$ is C^k, $k\geq 1$, and $\frac{d}{dt}\alpha(t,x) = f(t, \alpha(t,x))$, $\alpha(t_0, x) = x$, then α is also C^k.

defined in an open nbhd of $\{t_0\}\times \overset{\text{open}}{B}(x_0, b)$

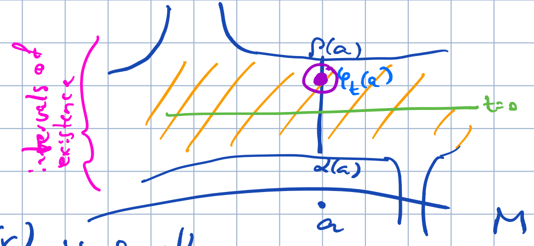
(case $k = \infty$: $f\in C^\infty \Rightarrow \alpha$ depends smoothly on init. cond.)

Proof of THM @ By THM*, $\forall a\in M$ we have an interval $(\alpha(a), \beta(a))$ on which maximal integral curve is defined. P-L THM (local existence) + continuous dependence on init. cond. x_0 (THM 10.5 in Hitchin) also implies that there is a solution for init. conditions in a nbhd of a.

So, the set

$$V = \{(t,x) \in \mathbb{R} \times M : t \in (\alpha(x), \beta(x))\}$$ is open.

- This the set on which $\varphi_t(x)$ is maximally defined



① Smooth dependence thm** says that $(t,x) \longmapsto \varphi_t(x)$ is smooth.

② Consider $\varphi_t \circ \varphi_s(x)$. fix s, vary t $\rightsquigarrow$ this is the unique int. curve of X through $\varphi_s(x)$

but $\varphi_{t+s}(x)$ is also an integral curve passing through $\varphi_s(x)$ at $t=0 \Longrightarrow \varphi_t \circ \varphi_s(x) = \varphi_{t+s}(x)$

Also: $\varphi_t \circ \varphi_{-t} = \mathrm{Id}$ $\Rightarrow$ we have a diffeo wherever it is defined.

③ Consider the case when M is compact.

$\forall x \in M$ we have an open interval $(\tilde{\alpha}(x), \tilde{\beta}(x))$ (possibly non-max. interval) containing 0 and an open nbhd $U_x \subset M$

s.t. $\varphi_t(y)$ is defined on $(\tilde{\alpha}(x), \tilde{\beta}(x)) \times U_x$. Cover M by $\{U_x\}_{x \in M}$ and

take a finite subcovering $U_{x_1}, \dots, U_{x_N}$; set $I = \bigcap_{i=1}^{N} (\tilde{\alpha}(x_i), \tilde{\beta}(x_i))$ - open interval containing 0.

Thus, for $t \in I$, we have $\varphi_t : I \times M \to M$,

defining (possibly non-maximal) int. curve through each $x \in M$, with $\varphi_0(x) = x$.

We need to extend to all real values t.

- define $\boxed{\varphi_t = (\varphi_{t/n})^n}$ multiplication = composition

  where n is large enough, so that $\frac{t}{n} \in I$.

  - this is well-defined, if we choose m s.t. $\frac{t}{m} \in I$, then

$$(\varphi_{t/m})^m = \left(\left(\varphi_{\frac{t}{m\cdot n}}\right)^n\right)^m = \left(\left(\varphi_{\frac{t}{m\cdot n}}\right)^m\right)^n = (\varphi_{t/n})^n$$

$$= \left(\varphi_{\frac{t}{m\cdot n}}\right)^{n\cdot m} //$$

  - compatible with composition:

$$\varphi_t \circ \varphi_s = \left(\varphi_{\frac{t}{n}}\right)^n \left(\varphi_{\frac{s}{n}}\right)^n = \left(\varphi_{\frac{t+s}{n}}\right)^n = \varphi_{t+s}$$

choose n s.t. $\frac{t+s}{n}, \frac{t}{n}, \frac{s}{n} \in I$

$$\varphi_{t/n} \circ \varphi_{s/n} = \varphi_{\frac{t+s}{n}} = \varphi_{s/n} \circ \varphi_{t/n}$$ - commute!

☐

Lie bracket revisited

functions, vector fields, [other objects] can be naturally transformed by diffeos ⇝ by "infinitesimal diffeos".

- natural action of diffeo $F: M \to M$ on functions $f \in C^\infty(M)$: $f \mapsto F^* f$

  ⇝ specialize to $F = \varphi_t$ a flow and take $\frac{d}{dt}\big|_{t=0}$ → we get a vector field

$$X: f \longmapsto \frac{d}{dt}\Big|_{t=0} \varphi_t^* f$$

WARNING: we can only do it for F a diffeo, not any smooth map!

- if Y is a vector field on M, $Y: M \to TM$, we can construct its "pullback" along F:

$$DF_x : T_xM \longrightarrow T_{F(x)}M$$
$$\tilde{Y}_x \longmapsto Y_{F(x)}$$

– this defines a new vector field $\tilde{Y}$ on M via $Y_{F(x)} = DF_x(\tilde{Y}_x)$ (#)

I.e. $\tilde{Y}_x(F^* f) = Y_{F(x)}(f)$ ← recall that $DF_a(X_a)(f) = X_a(F^* f)$

$\quad \tilde{Y}(F^* f)|_x \qquad F^* Y(f)|_x$

$$\Rightarrow \quad \tilde{Y}(F^* f) = F^* Y(f) \qquad (*)$$

Setting $F = \varphi_t$ a flow corresponding to a vector field X, we have $\tilde{Y}_t$

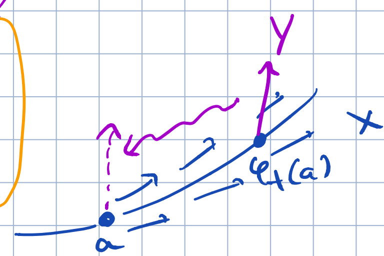
→ we can differentiate: $\dot{Y} = \frac{\partial}{\partial t}\Big|_{t=0} \tilde{Y}_t$

from (*): $\dot{Y}(f) + Y X(f) = X Y(f)$

So that $\dot{Y} = XY - YX$

(#)

put another way:
$$\boxed{[X,Y]_a = \frac{d}{dt}\Big|_{t=0} D\varphi^X_{-t}\left(Y_{\varphi^X_t(a)}\right)}$$



- We want to define "tensors" on manifolds generalizing functions and vector fields.
- We'll need a construction from linear algebra.

Tensor product of vector spaces

Let V, W vector spaces over $\mathbb{R}$. We will define a new v.sp. $V \otimes W$ equipped with a map

$$V \times W \longrightarrow V \otimes W$$
$$(v, w) \longmapsto v \otimes w$$

which is bilinear:

$$(\lambda v_1 + \mu v_2) \otimes w = \lambda v_1 \otimes w + \mu v_2 \otimes w$$
$$v \otimes (\lambda w_1 + \mu w_2) = \lambda v \otimes w_1 + \mu v \otimes w_2$$

Universal property of the tensor product:

If $B : V \times W \to U$ is a bilinear map to a v.sp. U,

then there is a unique linear map $\beta : V \otimes W \longrightarrow U$ such that $\boxed{B(v,w) = \beta(v \otimes w)}$

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "B"'] & V \otimes W \arrow[d, dashed, "\exists!\, \beta"] \\ & U \end{tikzcd}$$

- construction for V, W finite-dimensional:

$V \otimes W :=$ dual space to the space of bilinear forms $B : V \times W \to \mathbb{R}$

for $v \in V$, $w \in W$, we set $v \otimes w = (B \longmapsto B(v,w)) \in V \otimes W$

It satisfies the Univ. property: if $B' : V \times W \to U$,

$$(\xi \in U^*) \xmapsto{\ \varphi\ } (\xi \circ B' : V \times W \to \mathbb{R})$$

is a lin. map $U^* \to$ Bilinear forms on V, W

the dual map is $\beta : V \otimes W \longrightarrow (U^*)^* = U$

$$(B \longmapsto B(v,w)) \longmapsto \Big(\underset{U^*}{\xi} \longmapsto \underbrace{\langle v \otimes w, \xi \circ B' \rangle}_{= \xi \circ B'(v,w)} \Big)$$

$$\left(\begin{array}{l} X \xrightarrow{\varphi} Y \text{ lin map} \\ Y^* \xrightarrow{\varphi^*} X^* \\ \xi \longmapsto (x \longmapsto \langle \xi, \varphi(x) \rangle) \end{array} \right)$$

$$\begin{tikzcd} v \times w \arrow[r, mapsto, "\otimes"] \arrow[dr, "B'"'] & v \otimes w = (B \mapsto B(v,w)) \arrow[d] \\ & B'(v,w) \end{tikzcd}$$

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aside: abstract/general construction:

$V \otimes W := F(V \times W)/\sim$, $\sim$ - equiv. rel. gen. by $(v,w) + (v',w) \sim (v+v', w)$, $(v,w) + (v,w') \sim (v, w+w')$, $c(v,w) \sim (cv, w) \sim (v, cw)$

free vector space on a set: $F(V \times W) = \{ \sum_i c_i (v_i, w_i) \}$ - finite sums.

$v \otimes w := [(v,w)]$

- If $v_1, \dots, v_m$ – basis in V, $w_1 \dots w_n$ – basis in W, then a bilinear form B is uniquely determined by values $B(v_i, w_j)$.

Thus, $\boxed{\dim(V \otimes W) = (\dim V)(\dim W)}$

- vectors $\{v_i \otimes w_j\}_{\substack{i=1\dots m \\ j=1\dots n}}$ form a basis in $V \otimes W$

i.e. elements of $V \otimes W$ have the form $\sum_{i,j} a_{ij} v_i \otimes w_j$ (they are generally not pure product $v \otimes w$!)

(• assume $\sum a_{ij} v_i \otimes w_j = 0$, i.e. $\left(\sum a_{ij} v_i \otimes w_j\right)(B) = 0 \ \forall B \in \mathrm{Bil}$
$\sum a_{ij} B(v_i, w_j)$ $\Rightarrow$ all $a_{ij} = 0$ $\Rightarrow$ $v_i \otimes w_j$ are lin. indep.)

- We can form tensor powers of V: $V \otimes V = V^{\otimes 2}$, $V \otimes V \otimes V = V^{\otimes 3}, \dots$

$$V^{\otimes p} = (\text{space } p\text{-fold multilinear forms on } V)^*$$

tensor algebra: $TV := \bigoplus_{k=0}^{\infty} V^{\otimes k}$

an element is a <u>finite</u> sum of products of vectors $v_i \in V$: $\lambda \cdot 1 + v_0 + \sum_{i,j} v_i \otimes v_j + \dots + \sum_{i_1 \dots i_p} v_{i_1} \otimes \dots \otimes v_{i_p}$

multiplication on TV – extension by linearity of the product
$$(v_1 \otimes \dots \otimes v_p)(u_1 \otimes \dots \otimes u_q) = v_1 \otimes \dots \otimes v_p \otimes u_1 \otimes \dots \otimes u_q$$

– it is associative but not commutative.

<u>Exterior algebra</u>

TV – tensor algebra of a v.sp. V. Let $I(V)$ = ideal generated by elements $\boxed{v \otimes v}$, for $v \in V$

I.e. $I(V) = \left\{ \sum_i \alpha_i (v_i \otimes v_i) \beta_i \;\middle|\; \begin{array}{l} v_i \in V \\ \alpha_i, \beta_i \in T(V) \end{array} \right\}$

<u>def</u> The <u>exterior algebra</u> of V is the quotient $\Lambda^{\bullet} V := TV / I(V)$.

If $\pi: TV \to \Lambda^{\bullet} V$ the quotient map,

$\Lambda^p V := \pi(V^{\otimes p})$ – "p-fold exterior power of V"

$= \Big($multilinear forms $M(v_1, \dots, v_p)$ on V which vanish if any two arguments coincide$\Big)^*$

= "alternating multilinear forms"

<u>def</u> The <u>exterior product</u> of $\alpha = \pi(a) \in \Lambda^p V$ and $\beta = \pi(b) \in \Lambda^q V$ is $\alpha \wedge \beta = \pi(a \otimes b)$.

Rem: for $v_1, \dots, v_p \in V$, we define the element in $(\text{alternating forms})^*$ by

$$v_1 \wedge \dots \wedge v_p (M) := M(v_1, \dots, v_p)$$