LAST TIME:

- X - vector field $\rightsquigarrow$ $\varphi : U \to M$ - (possibly, non-global) flow, integrating X

  $U \subset \mathbb{R} \times M$ $\varphi_{t+s} = \varphi_t \circ \varphi_s$, $D\varphi_{(t,x)}\left(\frac{d}{dt}\right) = X_{\varphi(t,x)}$

- if M compact, a global flow $\varphi : \mathbb{R} \times M \to M$ exists

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"Geometric" formula for the Lie bracket of vector fields X, Y:

$$[X,Y]_a = \frac{d}{dt}\Big|_{t=0} \left(D\varphi^X_{-t}\right)_{\varphi^X_t(a)} \left(Y_{\varphi^X_t(a)}\right)$$

- We want to define "tensors" on manifolds generalizing functions and vector fields.
- We'll need a construction from linear algebra.

Tensor product of vector spaces

Let V, W vector spaces over $\mathbb{R}$. We will define a new v.sp. $V \otimes W$ equipped with a map

$$V \times W \longrightarrow V \otimes W \qquad (v, w) \longmapsto v \otimes w$$

which is bilinear:

$$(\lambda v_1 + \mu v_2) \otimes w = \lambda v_1 \otimes w + \mu v_2 \otimes w$$
$$v \otimes (\lambda w_1 + \mu w_2) = \lambda v \otimes w_1 + \mu v \otimes w_2$$

Universal property of the tensor product:

If $B: V \times W \to U$ is a bilinear map to a v.sp. U, then there is a unique linear map $\beta: V \otimes W \longrightarrow U$ such that $\boxed{B(v,w) = \beta(v \otimes w)}$

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "B"'] & V \otimes W \arrow[d, dashed, "\exists! \beta"] \\ & U \end{tikzcd}$$

- Construction for V, W finite-dimensional:

$V \otimes W :=$ dual space to the space of bilinear forms $B: V \times W \to \mathbb{R}$

for $v \in V$, $w \in W$, we set $v \otimes w = (B \longmapsto B(v,w)) \in V \otimes W$

It satisfies the Univ. property: if $B': V \times W \to U$,

$$(\xi \in U^*) \xrightarrow[\varphi]{} (\xi \circ B' : V \times W \to \mathbb{R})$$

is a lin. map $U^* \to$ Bilinear forms on V, W

the dual map is $\beta: V \otimes W \longrightarrow (U^*)^* = U$

$$\left(\begin{array}{l} X \xrightarrow{\varphi} Y \text{ lin. map} \\ Y^* \xrightarrow{\varphi^*} X^* \\ \gamma \longmapsto (x \mapsto \langle \gamma, \varphi(x) \rangle) \end{array} \right)$$

$$(B \longmapsto B(v,w)) \longmapsto \Big(\underset{\in U^*}{\xi} \longmapsto \underset{= \xi \circ B'(v,w)}{\langle v \otimes w, \xi \circ B' \rangle} \Big)$$

$$\begin{tikzcd} v \times w \arrow[r, mapsto, "\otimes"] \arrow[d, "B'"'] & v \otimes w = (B \mapsto B(v,w)) \arrow[dl] \\ B'(v,w) & \end{tikzcd}$$

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aside | abstract/general construction:

$V \otimes W := F(V \times W)/\sim$, $\sim$ - equiv. rel. gen. by $(v,w) + (v',w) \sim (v+v', w)$, $(v,w) + (v,w') \sim (v, w+w')$, $c(v,w) \sim (cv, w) \sim (v, cw)$

free vector space on a set: $F(V \times W) = \{ \sum_i c_i (v_i, w_i) \}$ - finite sums.

$v \otimes w := [(v,w)]$

- If $v_1,\dots,v_m$ - basis in V, $w_1 \dots w_n$ - basis in W, then a bilinear form B is uniquely determined by values $B(v_i,w_j)$.

Thus, $\boxed{\dim(V\otimes W) = (\dim V)(\dim W)}$

- vectors $\{v_i \otimes w_j\}_{\substack{i=1\dots m\\ j=1\dots n}}$ form a basis in $V\otimes W$

i.e. elements of $V\otimes W$ have the form $\sum_{i,j} a_{ij}\, v_i\otimes w_j$ (they are generally not pure product $v\otimes w$!)

(• assume $\sum a_{ij} v_i\otimes w_j = 0$, i.e. $\underbrace{(\sum a_{ij} v_i\otimes w_j)(B)}_{\sum a_{ij} B(v_i,w_j)} = 0 \ \forall B \in \mathrm{Bil}$ $\Rightarrow$ all $a_{ij}=0$ $\Rightarrow$ $v_i\otimes w_j$ are lin. indep.)

- We can form tensor powers of V: $V\otimes V = V^{\otimes 2}$, $V\otimes V\otimes V = V^{\otimes 3}$, ...

$$V^{\otimes p} = (\text{space } p\text{-fold multilinear forms on } V)^*$$

tensor algebra: $TV := \bigoplus_{k=0}^{\infty} V^{\otimes k}$

an element is a finite sum of products of vectors $v_i \in V$: $\lambda\cdot 1 + v_0 + \sum_{i,j} v_i\otimes v_j + \dots + \sum_{i_1\dots i_p} v_{i_1}\otimes\dots\otimes v_{i_p}$

multiplication on TV - extension by linearity of the product

$$(v_1\otimes\dots\otimes v_p)(u_1\otimes\dots\otimes u_q) = v_1\otimes\dots\otimes v_p\otimes u_1\otimes\dots\otimes u_q$$

- it is associative but not commutative.

Exterior algebra

TV - tensor algebra of a v.sp. V. Let $I(V)$ = ideal generated by elements $\boxed{v\otimes v}$, for $v\in V$

I.e. $I(V) = \{ \sum_i \alpha_i (v_i\otimes v_i)\beta_i \mid v_i \in V,\ \alpha_i,\beta_i \in T(V) \}$

def The exterior algebra of V is the quotient $\Lambda^\bullet V := TV / I(V)$.

If $\pi: TV \to \Lambda^\bullet V$ the quotient map,

$\Lambda^p V := \pi(V^{\otimes p})$ - "p-fold exterior power of V"

$= ($multilinear forms $M(v_1,\dots,v_p)$ on V which vanish if any two arguments coincide$)^*$

= "alternating multilinear forms" ($\longleftrightarrow$ M is anti-symmetric in its p arguments)

def The exterior product of $\alpha = \pi(a) \in \Lambda^p V$ and $\beta = \pi(b) \in \Lambda^q V$ is $\alpha\wedge\beta = \pi(a\otimes b)$.

Rem: for $v_1, \dots, v_p \in V$, we define the element in (alternating forms)* by

$v_1 \wedge \dots \wedge v_p (M) := M(v_1, \dots, v_p)$

Proposition 5.2 (#) If $\alpha \in \Lambda^p V$, $\beta \in \Lambda^q V$ then $\boxed{\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha}$ (*)

Proof: $\forall v \in V$, $v \otimes v \in I(V) \Rightarrow v \wedge v = 0 \Rightarrow$

$0 = (v_1 + v_2) \wedge (v_1 + v_2) = 0 + v_1 \wedge v_2 + v_2 \wedge v_1 + 0 \Rightarrow v_1 \wedge v_2 = -v_2 \wedge v_1$

$\Rightarrow$ interchanging two entries from V in $v_1 \wedge \dots \wedge v_k$ changes the sign

$\rightsquigarrow$ it suffices to check (*) for $\alpha = v_1 \wedge \dots \wedge v_p$, $\beta = w_1 \wedge \dots \wedge w_q$.
It then follows for general α, β by linearity in α and β.

- $(v_1 \wedge \dots \wedge v_p) \wedge (w_1 \wedge \dots \wedge w_q) = (-1)^p \, w_1 \wedge \underbrace{(v_1 \wedge \dots \wedge v_p)}_{(-1)^p \text{ - similarly}} \wedge (w_2 \wedge \dots \wedge w_q)$ ← bring w_1 to the left.

$= (-1)^{2p} (w_1 \wedge w_2) \wedge (v_1 \wedge \dots \wedge v_p) \wedge (w_3 \wedge \dots \wedge w_q)$ ← a factor of $(-1)^p$ for bringing each of w_i in front.

$\dots = (-1)^{pq} (w_1 \wedge \dots \wedge w_q) \wedge (v_1 \wedge \dots \wedge v_p)$

$\Rightarrow \alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$ ☐

(#)

Proposition 5.3 If $\dim V = n$, then $\dim \Lambda^n V = 1$

Proof:
- For $w_1, \dots, w_n \in V$, consider $B(w_1, \dots, w_n) := \det M$ — a non-zero alternating n-linear form on V

M: matrix with columns = coordinate vectors of $w_1, \dots, w_n$ rel. to some basis in V

So, $B \in (\Lambda^n V)^*$, $B \neq 0$, so $\dim \Lambda^n V \geq 1$

- if $v_1, \dots, v_n$ - basis in V, then anything in $V^{\otimes n}$ is a lin. comb. of $v_{i_1} \otimes \dots \otimes v_{i_n}$.
So, by Prop. (#), anything in $\Lambda^n V$ is a lin. comb. of $v_1 \wedge \dots \wedge v_n$.
so, $\dim \Lambda^n V \leq 1$ ☐

Proposition 5.4 Let $v_1, \dots, v_n$ be a basis for V. Then the $\binom{n}{p}$ elements $\{v_{i_1} \wedge \dots \wedge v_{i_p}\}_{1 \leq i_1 < \dots < i_p \leq n}$ form a basis for $\Lambda^p V$.

Proof $\Lambda^p V$ is spanned by $\{v_{i_1} \wedge \dots \wedge v_{i_p}\}_{1 \leq i_1 \leq n, \dots, 1 \leq i_p \leq n} \underset{\text{reordering / changing sign}}{\Rightarrow} \Lambda^p V = \mathrm{Span}\{v_{i_1} \wedge \dots \wedge v_{i_p}\}_{i_1 < \dots < i_p}$

Linear independence: suppose $\sum_{i_1<\dots<i_p} a_{i_1\dots i_p} v_{i_1}\wedge\dots\wedge v_{i_p} = 0$ (@)

$$@ \wedge \overbrace{\boxed{v_{j_1}\wedge\dots\wedge v_{j_{n-p}}}}^{\beta} \Rightarrow \pm a_{i_1\dots i_p} \underbrace{v_1\wedge\dots\wedge v_n}_{\neq 0} = 0$$

where $\underbrace{\{i_1,\dots,i_p\}}_{\text{fixed subset of } \{1\dots n\}} \cup \{j_1,\dots,j_{n-p}\} = \{1,2,\dots,n\}$

(all other terms in the lhs of @ vanish after wedging with β)

$\Rightarrow a_{i_1\dots i_p} = 0$ for any $i_1<\dots<i_p$ $\square$

Proposition 5.5 v is linearly dependent on the set of lin. indep. vectors $v_1,\dots,v_p$

iff $v_1\wedge\dots\wedge v_p\wedge v = 0$

Proof: if $v=\sum_i a_i v_i$ - lin. dependent on v_i s, then

$(v_1\wedge\dots\wedge v_p)\wedge v = \sum_i a_i v_1\wedge\dots\wedge v_p\wedge v_i = 0$

if v is lin. indep., then the set $v_1,\dots,v_p,v$ can be extended into a basis for V and by the previous Prop. 5.4, $v_1\wedge\dots\wedge v_p\wedge v\neq 0$. $\square$

def if $A: V\to W$ is a lin. transformation, then there is an induced linear transformation $\Lambda^p A: \Lambda^p V \to \Lambda^p W$ defined by $v_1\wedge\dots\wedge v_p \mapsto Av_1\wedge\dots\wedge Av_p$ (and extended by linearity)

• Another construction of $\Lambda^p A$: by univ. property

$$\begin{tikzcd} V\times V \arrow[r, "\otimes"] \arrow[dr] & V\otimes V \arrow[d, dashed, "A^{\otimes 2}"] \\ & W\otimes W \end{tikzcd} \qquad (v_1,v_2)\mapsto Av_1\otimes Av_2$$

$$\cdots\cdots \quad \begin{tikzcd} V\times\dots\times V \arrow[r, "\otimes"] \arrow[dr] & V^{\otimes p} \arrow[d, dashed, "A^{\otimes p}"] \\ & W^{\otimes p} \end{tikzcd} \qquad (v_1,\dots,v_p)\mapsto Av_1\otimes\dots\otimes Av_p$$

• $A^{\otimes p}(I(V)) \subset I(W)$

$\Rightarrow$ $A^{\otimes p}$ induces a map of quotients $\Lambda^p A : \underbrace{V^{\otimes p}/I(V)}_{=\Lambda^p V} \longrightarrow \underbrace{W^{\otimes p}/I(W)}_{=\Lambda^p W}$

Proposition 5.7 If $\dim V = n$ and $A : V \to V$, then the linear transformation $\Lambda^n V \to \Lambda^n V$ is given by (multiplication by) $\det A$. ← determinant of the matrix of A rel. to some basis

Proof: $\Lambda^n V$ is 1-dimensional, so $\Lambda^n A : \Lambda^n V \to \Lambda^n V$ is a multiplication by a real number $\lambda(A)$. For $v_1, \dots, v_n$ a basis in V,

$$\Lambda^n A(v_1 \wedge \dots \wedge v_n) = \underbrace{Av_1 \wedge \dots \wedge Av_n}_{} = \lambda(A)\, v_1 \wedge \dots \wedge v_n$$

$$Av_i = \sum_j A_{ji} v_j$$

$$Av_1 \wedge \dots \wedge Av_n = \sum_{j_1 \dots j_n} A_{j_1 1} v_{j_1} \wedge \dots \wedge A_{j_n n} v_{j_n}$$

$$= \sum_{\sigma \in S_n} A_{\sigma(1) 1} v_{\sigma(1)} \wedge \dots \wedge A_{\sigma(n) n} v_{\sigma(n)}$$

permutations → $\sigma \in S_n$

$$= \underbrace{\sum_{\sigma \in S_n} \operatorname{sign} \sigma \cdot A_{\sigma(1) 1} \cdots A_{\sigma(n) n}}_{\det A}\; v_1 \wedge \dots \wedge v_n$$

$\square$

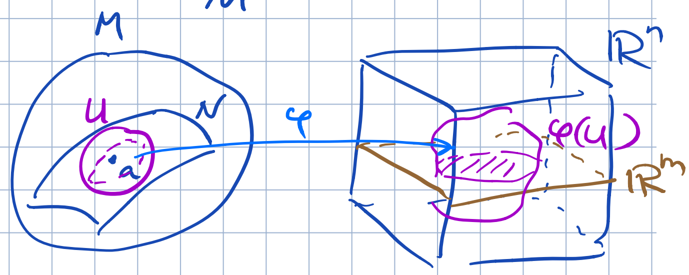
EXTRA

Submanifolds revisited

Alternative definition$^{(\alpha)}$ (see e.g. S. Stolz's notes)

$\underset{}{N} \subset \underbrace{M}_{n\text{-manifold}}$ is an m-dimensional submanifold if $\forall a \in N$ there is a chart $\underset{\substack{\cap \text{ open} \\ M}}{\overset{\substack{a \\ \cap}}{U}} \xrightarrow{\varphi} \mathbb{R}^n$

s.t. $\varphi(U \cap N) = \{(x_1, \ldots, x_m, 0, \ldots, 0) \in \varphi(U)\}$
$= \mathbb{R}^m \times \{0\} \cap \varphi(U)$



[(U, φ) with this property – "submanifold chart"]

- $x_1 \ldots x_m$ – loc coords. on N

Recall the old definition$^{(\beta)}$: A submanifold of M is $N = \iota(\tilde{N}) \subset M$

where $\iota : \tilde{N} \to M$ is an injective smooth map with $D\iota_a$ injective (immersion) $\forall a \in \tilde{N}$

and with manifold topology on $\tilde{N}$ being the induced (subspace) topology from M.

⟨ i.e. ι is a homeomorphism between $\tilde{N}$ and its image N ⟩

- A submanifold$^{(\alpha)}$ gives a submanifold$^{(\beta)}$ with $\tilde{N} = N$, ι = inclusion map

$(\beta) \Rightarrow (\alpha)$ is based on "local immersion theorem" (Thm 3.2 in A. Putman's notes)