Exterior derivative

Theorem On any smooth manifold M there is a natural linear map

$d: \Omega^p(M) \to \Omega^{p+1}(M)$ - the "exterior derivative" - such that

1. if $f \in \Omega^0(M)$ then $df \in \Omega^1(M)$ is the derivative of f
2. $d^2 = 0$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ if $\alpha \in \Omega^p(M)$.

Proof: locally: $\alpha := \sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_p}$

set $d\alpha = \sum_{i_1 < \dots < i_p} da_{i_1 \dots i_p} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}$ (##) - at $p=0$, this is the usual derivative
$\Rightarrow$ ① holds ✓

②: expand: $d\alpha = \sum_{j, i_1 < \dots < i_p} \frac{\partial a_{i_1 \dots i_p}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \dots dx_{i_p}$

$\Rightarrow \quad d^2\alpha = \sum_{j,k,\, i_1 < \dots < i_p} \underbrace{\frac{\partial^2 a_{i_1 \dots i_p}}{\partial x_j \partial x_k}}_{\text{symmetric in } j \rightleftarrows k} \underbrace{dx_k \wedge dx_j}_{\text{anti-sym in } j \rightleftarrows k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p} = 0$ ✓

(3) check on decomposable forms $\alpha = f\, dx_{i_1} \wedge \dots \wedge dx_{i_p} = f\, dx_I$ multi-indices and extend by linearity
$\beta = g\, dx_{j_1} \wedge \dots \wedge dx_{j_q} = g\, dx_J$

$$d(\alpha\wedge\beta) = d(fg\, dx_I \wedge dx_J)$$
$$= d(fg)\wedge dx_I \wedge dx_J$$
$$= (f\,dg + g\,df)\wedge dx_I \wedge dx_J$$
$$= (-1)^p \underbrace{(f\,dx_I)}_{\alpha}\wedge \underbrace{(dg\wedge dx_J)}_{d\beta} + \underbrace{(df\wedge dx_I)}_{d\alpha}\wedge \underbrace{(g\,dx_J)}_{\beta} \qquad ✓$$

- we defined d_x locally, using a coord. system satisfying ①,②,③. Using another coord. system:

$$\alpha = \sum_{i_1<\dots<i_p} b_{i_1\dots i_p}\, dy_{i_1}\wedge\dots\wedge dy_{i_p}, \qquad d'\alpha = \sum_{i_1<\dots<i_p} db_{i_1\dots i_p}\wedge dy_{i_1}\wedge\dots\wedge dy_{i_p}$$

d' defined same way but in y-coordinates

we'll prove $d = d'$ from ①,②,③:

$$d\alpha = d\left(\sum b_{i_1\dots i_p}\, dy_{i_1}\wedge\dots\wedge dy_{i_p}\right) \overset{(3),(1)}{=} \sum db_{i_1\dots i_p}\, dy_{i_1}\wedge\dots\wedge dy_{i_p} + b_{i_1\dots i_p} \underbrace{d(dy_{i_1}\wedge\dots\wedge dy_{i_p})}_{(3)}$$

$$\underbrace{d^2 y_{i_1}}_{\overset{(2)}{0}}\wedge\dots - dy_{i_1}\wedge d(dy_{i_2}\wedge\dots\wedge dy_{i_p}) = \dots = 0$$

$$= \sum db_{i_1\dots i_p}\, dy_{i_1}\wedge\dots\wedge dy_{i_p} \qquad = d'\alpha$$

So, on each coord. nbhd, d is given by (##) and is globally well-defined. □

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* **Coordinate-free definition of exterior derivative**

for $\alpha\in\Omega^p(M)$, $X_0,\dots,X_p$ - vector fields, $d\alpha$ is characterized by

$$(d\alpha)(X_0,\dots,X_p) = \sum_{i=0}^{p} (-1)^i X_i\, \alpha(X_0,\dots,\hat{X}_i,\dots,X_p) + \sum_{0\le i<j\le p} (-1)^{i+j}\alpha([X_i,X_j],X_0,\dots\hat{X}_i\dots\hat{X}_j\dots,X_p)$$

Ex: $p=0$ $\quad (df)(X) = X(f)$

$p=1$ $\quad (d\alpha)(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$

Proposition Let $F: M \to N$ be a smooth map and $\alpha \in \Omega^p(N)$. Then $d(F^*\alpha) = F^*(d\alpha)$.

Proof: We already know that $F^*(df) = d(F^*f)$. using $F^*(\beta \wedge \gamma) = F^*\beta \wedge F^*\gamma$

$\Rightarrow$ if $\alpha = \sum_{i_1<\dots<i_p} a_{i_1\dots i_p}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_p}$ then $F^*\alpha = \sum_{i_1<\dots<i_p} (F^* a_{i_1\dots i_p})(x)\, F^*dx_{i_1} \wedge \dots \wedge F^*dx_{i_p}$

$$\Rightarrow d(F^*\alpha) = \sum_{i_1<\dots<i_p} dF^* a_{i_1\dots i_p} \wedge F^*dx_{i_1} \wedge \dots \wedge F^*dx_{i_p}$$
$$= \sum_{i_1<\dots<i_p} F^* da_{i_1\dots i_p} \wedge F^*dx_{i_1} \wedge \dots \wedge F^*dx_{i_p}$$
$$= F^* d\alpha \qquad \square$$

Lie derivative of a differential form

def Let X be a vector field on a manifold M and $\alpha \in \Omega^p(M)$ a p-form. **Lie derivative** of α along X is defined as

$$\mathcal{L}_X \alpha := \frac{d}{dt}\Big|_{t=0} \varphi_t^* \alpha$$

where $\varphi_t : \underset{\mathbb{R}\times M}{\overset{\cap}{U}} \to M$ - (local) flow on M generated by X.

Proposition: Given a vector field X on M, there is a linear map

$\iota_X : \Omega^p(M) \to \Omega^{p-1}(M)$ (the inner product, or contraction with X, or substitution of X)

such that

(i) $\iota_X df = X(f)$

(ii) $\iota_X(\alpha \wedge \beta) = \iota_X \alpha \wedge \beta + (-1)^p \alpha \wedge \iota_X \beta$ if $\alpha \in \Omega^p$

$\underset{p=0}{\Rightarrow} \iota_X(f\alpha) = f \cdot \iota_X \alpha$

Ex: if $X = \sum_i a_i \frac{\partial}{\partial x_i}$, $\alpha = f dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p}$ then

$$\iota_X \alpha \underset{(i),(ii)}{=} f a_{i_1} dx_{i_2} \wedge \dots \wedge dx_{i_p} - f a_{i_2} dx_{i_1} \wedge dx_{i_3} \wedge \dots \wedge dx_{i_p} + \dots \qquad (\#)$$

$\Rightarrow \iota_X(\iota_X \alpha) = f a_{i_1} a_{i_2} dx_{i_3} \wedge \dots \wedge dx_{i_p} - f a_{i_2} a_{i_1} dx_{i_3} \wedge \dots \wedge dx_{i_p} + \dots = 0$

Ex $\alpha = dx \wedge dy$ $\quad X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ $\quad \Rightarrow \iota_X \alpha = x\,dy - y\,dx$.

Proof: $\Lambda^p V = \big(\text{alternating } p\text{-multilinear forms } \mu : \underbrace{V\times\dots\times V}_{p} \to \mathbb{R}\big)^* = \mathrm{Alt}^p(V)^*$

If $\mu : \underbrace{V\times\dots\times V}_{p-1} \to \mathbb{R}$ alt. multilin. form and $\xi \in V^*$, then

$(\xi\mu)(v_1,\dots,v_p) = \xi(v_1)\mu(v_2,\dots,v_p) - \xi(v_2)\mu(v_1,v_3,\dots,v_p) + \dots$ (##) - an alternating p-multilin. form on V

If $\alpha \in \Lambda^p V$, define $(\iota_\xi \alpha)(\mu) := \alpha(\xi\mu)$.

Taking $V = T_x^* M$, $\xi = X_x \in V^* = T_x M$, we get the interior product;

(##) $\Rightarrow$ (#) $\leadsto$ can compute the interior product.

$\boxed{?\square}$

Alternative (equivalent) definition of $\iota_X \alpha$.

$\alpha \in \Omega^p(M)$ can be seen [by a HW problem] as a map

$\mathfrak{X}(M)$ = space of vector fields on M.

$$\alpha : \mathfrak{X}(M)\times\dots\times\mathfrak{X}(M) \longrightarrow C^\infty(M)$$
$$(X_1,\dots,X_p) \longmapsto \alpha(X_1,\dots,X_p)$$

which is
- skew-symmetric
- $C^\infty(M)$-linear in each argument

Then: $\boxed{(\iota_X \alpha)(X_1,\dots,X_{p-1}) := \alpha(X, X_1,\dots,X_{p-1})}$

- a $(p-1)$-form.

check: $p=1$ $\quad \iota_X df = df(X) = X(f)$

Proposition For α a p-form on M and X a vector field, the Lie derivative is:
$$\mathcal{L}_X \alpha = d(\iota_X \alpha) + \iota_X(d\alpha)$$

Proof Denote rhs $= R_X(\alpha) = d\,\iota_X \alpha + \iota_X d\alpha$. • R_X maps p-forms to p-forms

We have: • $R_X(d\alpha) = d\,\iota_X d\alpha + \cancel{\iota_X dd\alpha} = d R_X(\alpha)$ ← R_X commutes with d.

• $R_X(\alpha \wedge \beta) = R_X\alpha \wedge \beta + \alpha \wedge R_X\beta$ – since $\iota_X(\alpha\wedge\beta) = \iota_X\alpha \wedge\beta + (-1)^p \alpha \wedge \iota_X\beta$
$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p \alpha\wedge d\beta$

On the other hand, $\varphi_t^*(d\alpha) = d\,\varphi_t^*(\alpha) \overset{\frac{d}{dt}|_{t=0}}{\Longrightarrow} \mathcal{L}_X d\alpha = d\,\mathcal{L}_X\alpha$

$$\varphi_t^*(\alpha\wedge\beta) = \varphi_t^*(\alpha)\wedge\varphi_t^*(\beta) \overset{\frac{d}{dt}|_{t=0}}{\Longrightarrow} \mathcal{L}_X(\alpha\wedge\beta) = \mathcal{L}_X\alpha\wedge\beta + \alpha\wedge\mathcal{L}_X\beta$$

So: $\mathcal{L}_X$ and R_X – preserve degree, commute with d, satisfy (same) Leibnitz identity

$\Rightarrow$ for $\alpha = \sum_{i_1 < \dots < i_p} \alpha_{i_1 \dots i_p}(x)\, dx_{i_1}\wedge\dots\wedge dx_{i_p}$ $\mathcal{L}_X$ and R_X agree if they agree on functions.

$$R_X f = \iota_X df = X(f) = \frac{d}{dt}\Big|_{t=0} f(\varphi_t) = \mathcal{L}_X(f)$$
by def. of a flow of X

$\square$