LAST TIME

tangent space

$$T_a \cong T_a^{Alg}$$

$$T_a = \underbrace{\left(C^\infty(M)/Z_a\right)^*}_{T_a^*}$$

$$T_a^{Alg} = \left\{ \text{tangent vectors } X_a : C^\infty(M) \to \mathbb{R} \;\middle|\; X_a(fg) = f(a)X_a(g) + g(a)X_a(f) \right\}$$

$$\xi \in T_a \longmapsto \left(X_a : f \longmapsto \xi(df_a)\right)$$

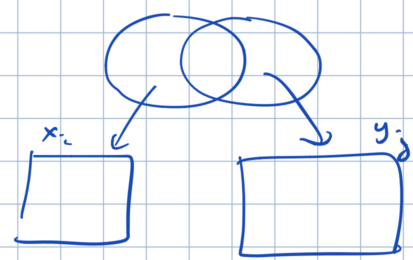
$$\left(\xi : (df)_a \mapsto X_a(f)\right) \longleftarrow X_a$$

derivative of a function locally:

$$(df)_a = \sum_i \left(\frac{\partial f}{\partial x_i}\right)_a \underbrace{(dx_i)_a}_{\text{basis vectors} \,\sim\, T_a^*}$$

$$\cancel{= \sum_j \Bigg(\underbrace{\sum_i \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial y_j}}_{\frac{\partial f}{\partial y_j}} \Bigg) dy_j}$$

$$(dx_i)_a = \sum_j \frac{\partial x_i}{\partial y_j} (dy_j)_a$$



$$\left(\frac{\partial}{\partial x_i}\right)_a = \sum_j \frac{\partial y_j}{\partial x_i} \left(\frac{\partial}{\partial y_j}\right)_a$$

(chain rule)

Thus, vectors in T_a are the tangent vectors

Locally; in coordinates: $X_a = \sum_{i=1}^n c_i \left(\frac{\partial}{\partial x_i}\right)_a$

then $$X_a f = \sum_{i=1}^n c_i \frac{\partial f}{\partial x_i}(a) \qquad (@)$$

Derivatives of smooth maps

Suppose $F : M \to N$ smooth map, $f \in C^\infty(N)$. Then $f \circ F \in C^\infty(M)$.

Rem: $C^\infty(N) \xrightarrow{F^*} C^\infty(M)$, $f \longmapsto f \circ F$ is a homomorphism of rings.

"$F^* f$" – "pullback of f along F"

②

def The *derivative* at $a \in M$ of the smooth map $F: M \to N$ is the homomorphism of tangent spaces $DF_a: T_aM \to T_{F(a)}N$ defined by
$$DF_a(X_a)(f) = X_a(f \circ F)$$

- This is an abstract, coord-free definition. In coordinates, using (@):
$$DF_a\left(\frac{\partial}{\partial x_i}\right)_a (f) = \frac{\partial}{\partial x_i}(f \circ F)(a) = \sum_j \frac{\partial F_j}{\partial x_i}(a)\frac{\partial f}{\partial y_j}(F(a)) = \sum_j \frac{\partial F_j}{\partial x_i}(a)\left(\frac{\partial}{\partial y_j}\right)_{F(a)} f$$
i.e. $DF_a : \left(\frac{\partial}{\partial x_i}\right)_a \longmapsto \sum_j \frac{\partial F_j}{\partial x_i}(a)\left(\frac{\partial}{\partial y_j}\right)_{F(a)}$

thus, DF_a is an invariant way of defining the Jacobian matrix.

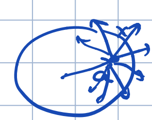
Thm Let $F: M \to N$ be a smooth map and $c \in N$ be such that for each $a \in F^{-1}(c)$, the derivative DF_a is surjective. (i.e. c is a "reg. value of F") Then $F^{-1}(c)$ is a smooth manifold of dimension $\dim M - \dim N$.

- inclusion $\iota: F^{-1}(c) \hookrightarrow M$ is a smooth map, $D\iota$ is injective and $\operatorname{im} D\iota_a = \ker DF_a$ [← exercise]

Thus: $T_aF^{-1}(c) \cong \ker DF_a$ — helps understand tangent spaces in the case $M = \mathbb{R}^n$.

Examples: 1) $S^n = F^{-1}(1)$, $F: \mathbb{R}^{n+1} \to \mathbb{R}$, $x \longmapsto \|x\|^2$ $\qquad DF_a(x) = \sum_i 2x_i a_i$

$\ker DF_a = \{\text{vectors orthogonal to } a \in S^n\}$



2) $O(n) = F^{-1}(I)$, $F: Mat_{n\times n} \to SymMat_{n\times n}$, $A \longmapsto A^TA$ $\qquad DF_I(H) = H^T + H$

$\ker DF_I = \{H \in Mat_{n\times n} \mid H^T = -H\} = \{\text{skew-sym. matrices}\}$.

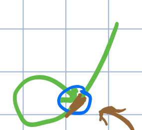
def An (embedded) submanifold of N is the *image* $\iota(M) \subset N$ of an inclusion map $\iota: M \to N$ s.t.

(a) ι is smooth

(b) $D\iota_x$ is injective for each $x \in M$

(c) the topology on M coincides with the induced (subspace) one from N.

→ to avoid a situation like



$(-1,\infty) \xrightarrow{\iota} \mathbb{R}^2$, $t \longmapsto (t^2-1, t(t^2-1))$ — part of the cubic $y^2 = x^2(x+1)$

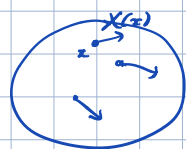
$(1-\delta, 1+\delta)$ not open in induced topology!

Vector fields

- The tangent bundle.

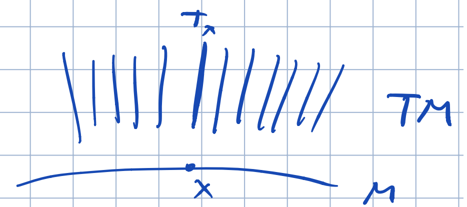
imagine: wind velocity at every point on Earth at a given time

- vector field on S^2 - smooth map $X: S^2 \to \mathbb{R}^3$ s.t. $X(x)$ is tangential to S^2 at x.



- want a general definition of a v.f. on M without a reference to any ambient space.

"Pre-definition"
a vector field on M is a family of vectors $X_a \in T_a$, $a \in M$ (one for each)
"varying smoothly as a moves on M."



- Let $TM = \bigcup_{x \in M} T_x$ (a set) - the disjoint union of all tangent spaces

Let (U, φ_u) be a coord. chart on M. For $x \in U$, $\left\{ \left(\frac{\partial}{\partial x_1}\right)_x, \dots, \left(\frac{\partial}{\partial x_n}\right)_x \right\}$ - basis for T_x (the tangent vectors)

$$\psi_u : U \times \mathbb{R}^n \longrightarrow \bigcup_{x \in U} T_x \qquad \text{- bijection}$$

$$(x, y_1, \dots, y_n) \longmapsto \sum_{i=1}^n y_i \left(\frac{\partial}{\partial x_i}\right)_x$$

Thus, $\Phi_u = (\varphi_u, id) \circ \psi_u^{-1} : \underbrace{\bigcup_{x \in U} T_x}_{V \subset TM} \longrightarrow \varphi_u(U) \times \mathbb{R}^n$ $\quad (*)$

is a coordinate chart for $V = \bigcup_{x \in U} T_x$.

(Topology on TM - generated by $\{ \Phi_u^{-1}(\text{open balls in } \varphi_u(U) \times \mathbb{R}^n) \}$)

(or: $W \subset TM$ is open if $\Phi_{u_\alpha}(W \cap V_\alpha)$ is open in $\mathbb{R}^{2n}$ $\forall\, U_\alpha$)

(· given U_α, U_β coord. charts on M,

$$\Phi_\alpha(V_\alpha \cap V_\beta) = \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$
- open in $\mathbb{R}^{2n}$)

· if $(x_1, \dots, x_n)$ - coords in U_α, $(\tilde{x}_1, \dots, \tilde{x}_n)$ - coords in $\tilde{U}_\beta$ then

$$\left(\frac{\partial}{\partial x_i}\right)_x = \sum_{j=1}^n \frac{\partial \tilde{x}_j}{\partial x_i} \left(\frac{\partial}{\partial \tilde{x}_j}\right)_x \quad \Rightarrow$$

$$\Phi_\beta \Phi_\alpha^{-1}(x_1, \dots, x_n; y_1, \dots, y_n) = \left(\tilde{x}_1, \dots, \tilde{x}_n; \sum_i \frac{\partial \tilde{x}_1}{\partial x_i} y_i, \dots, \sum_i \frac{\partial \tilde{x}_n}{\partial x_i} y_i\right)$$

- smooth in x, y (in fact, linear in y)

$\Leftarrow$ Jacobian $\left(\frac{\partial \tilde{x}_j}{\partial x_i}\right)$ is smooth in x

$\Rightarrow$ $(V_\alpha, \varphi_\alpha)$ defines an atlas on TM,

$$\boxed{\dim TM = 2n}$$

Def The <u>tangent bundle</u> of a manifold M is the $2n$-dimensional smooth manifold structure on TM defined by the atlas (V_α, Φ_α) above.

- The projection map
$$\begin{aligned} p : TM &\longrightarrow M \\ X_a \in T_a &\longmapsto a \end{aligned}$$
is <u>smooth</u>, with <u>surjective derivative</u>,
since in loc. coordinates it is given by $p(x_1,\dots,x_n;y_1,\dots,y_n) = (x_1,\dots,x_n)$

- $p^{-1}(a) = T_a$ the fiber of the projection

(Jacobi matrix of p in loc. coord: $\Big(\underbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}_{n} \Big| \underbrace{0}_{n}\Big)\Big\}n$ $\Rightarrow$ rank $= n$)

def A vector field on a manifold M is a smooth map $X : M \longrightarrow TM$ such that $p \circ X = id_M$ (- global definition)

Locally; since $p \circ X = id_M$, $X(x_1,\dots,x_n) = (x_1,\dots,x_n; y_1(x),\dots,y_n(x))$
where $y_i(x)$ - smooth functions

I.e., the tangent vector at x: $X(x) = \sum_{i=1}^{n} y_i(x) \left(\frac{\partial}{\partial x_i}\right)_x$ - smoothly varying field of tangent vectors.

Remark More generally, given a projection $p : Q \to M$, a <u>"section"</u> of p is a smooth map $s : M \to Q$ s.t. $p \circ s = id_M$.

- For $Q = TM$ the tangent bundle, we always have the <u>zero-section</u> - the vector field $X = 0$.

- Multiplying by a bump function μ, we can construct vector fields out of locally defined v.f.s $X(x) = \sum \underbrace{y_i(x)}_{\substack{\text{locally defined} \\ C^\infty \text{ functions}}} \left(\frac{\partial}{\partial x_i}\right)_x$

Remark Can similarly form the <u>cotangent bundle</u> $T^*M = \bigcup_a T_a^*$
using the basis $(dx_1)_x, \dots, (dx_n)_x$ in T_x^*
instead of the dual basis in T_x.

$$\lambda_U : U \times \mathbb{R}^n \longrightarrow \bigcup_{x \in U} T_x^*$$
$$(x_1 \dots x_n; z_1 \dots z_n) \longmapsto \sum z_i (dx_i)_x$$

$$\Phi_U = (\varphi_U, id) \circ \lambda_U^{-1} : \underbrace{\bigcup_{x \in U} T_x^*}_{W} \longrightarrow \varphi_U(U) \times \mathbb{R}^n$$

transition map: $\Psi_\beta \Psi_\alpha^{-1} : (x_1,\dots,x_n; z_1,\dots,z_n) \longmapsto (\tilde{x}_1,\dots,\tilde{x}_n; \sum_i \frac{\partial x_i}{\partial \tilde{x}_1} z_i, \dots, \sum_i \frac{\partial x_i}{\partial \tilde{x}_n} z_i)$

Then, the derivative of $f \in C^\infty(M)$ is a map $df : M \longrightarrow T^*M$ satisfying $p \circ df = id_M$. (but not every section of T^*M is a derivative!)

TM and T^*M are examples of vector bundles.

def A real vector bundle of rank m on a manifold M is a manifold E with a smooth projection map $p : E \longrightarrow M$ s.t.

- each fiber $p^{-1}(x)$ has the structure of an m-dimensional real vector space.
- each point $x \in M$ has a nbhd U and a diffeomorphism $\psi_U : p^{-1}(U) \xrightarrow{\simeq} U \times \mathbb{R}^m$ "local trivialization" s.t. $proj_1 \circ \psi_U = p$ and $proj_2 \circ \psi_U$ is a linear isomorphism from the v.sp. $p^{-1}(y)$ for any $y \in U$, to the v.space $\mathbb{R}^m$.

<consequence of the previous>

- on the intersection $U \cap V$,

$$\psi_U \circ \psi_V^{-1} : U \cap V \times \mathbb{R}^m \longrightarrow U \cap V \times \mathbb{R}^m$$

is of the form $(x, v) \longmapsto (x, g_{UV}(x) v)$

where $g_{UV}(x)$ is a smooth function on $U \cap V$ with values in invertible $m \times m$ matrices.

"transition function"

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for TM, g_{UV} is the Jacobian matrix $\left(\frac{\partial \tilde{x}_i}{\partial x_j}\right)$

for T^*M g_{UV} is its inverse transpose $\left(\frac{\partial x_j}{\partial \tilde{x}_i}\right)$