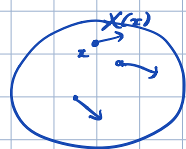
Vector fields

- The tangent bundle.

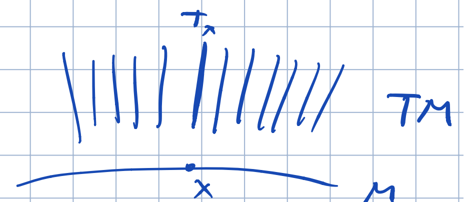
imagine: wind velocity at every point on Earth at a given time

- vector field on S^2 - smooth map $X: S^2 \to \mathbb{R}^3$ s.t. $X(x)$ is tangential to S^2 at x.



- want a general definition of a v.f. on M without a reference to any ambient space.

"Pre-definition"
a vector field on M is a family of vectors $X_a \in T_a$, $a \in M$ (one for each) "varying smoothly as a moves on M."



• Let $TM = \bigcup_{x \in M} T_x$ (a set) - the disjoint union of all tangent spaces

Let (U, φ_u) be a coord. chart on M. For $x \in U$, $\left\{ \left(\frac{\partial}{\partial x_1}\right)_x, \dots, \left(\frac{\partial}{\partial x_n}\right)_x \right\}$ - basis for T_x (the tangent vectors)

$$\psi_u : U \times \mathbb{R}^n \longrightarrow \bigcup_{x \in U} T_x \quad \text{- bijection}$$

$$(x, y_1, \dots, y_n) \longmapsto \sum_{i=1}^n y_i \left(\frac{\partial}{\partial x_i}\right)_x$$

Thus, $$\Phi_u = (\varphi_u, id) \circ \psi_u^{-1} : \underbrace{\bigcup_{x \in U} T_x}_{V \subset TM} \longrightarrow \varphi_u(U) \times \mathbb{R}^n \qquad (*)$$

is a coordinate chart for $V = \bigcup_{x \in U} T_x$.

(Topology on TM - generated by $\{ \Phi_u^{-1}(\text{open balls in } \varphi_u(U) \times \mathbb{R}^n) \}$)

(or: $W \subset TM$ is open if $\Phi_{u_\alpha}(W \cap V_\alpha)$ is open in $\mathbb{R}^{2n}$ $\forall\, U_\alpha$)

(• given U_α, U_β coord. charts on M,
$$\Phi_\alpha(V_\alpha \cap V_\beta) = \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$
- open in $\mathbb{R}^{2n}$)

• if $(x_1, \dots, x_n)$ - coords in U_α, $(\tilde{x}_1, \dots, \tilde{x}_n)$ - coords in U_β then

$$\left(\frac{\partial}{\partial x_i}\right)_x = \sum_{j=1}^n \frac{\partial \tilde{x}_j}{\partial x_i} \left(\frac{\partial}{\partial \tilde{x}_j}\right)_x \implies$$

$$\Phi_\beta \Phi_\alpha^{-1}(x_1, \dots, x_n; y_1, \dots, y_n) = \left(\tilde{x}_1, \dots, \tilde{x}_n; \sum_i \frac{\partial \tilde{x}_1}{\partial x_i} y_i, \dots, \sum_i \frac{\partial \tilde{x}_n}{\partial x_i} y_i\right)$$ - smooth in x, y (in fact, linear in y)

$\Leftarrow$ Jacobian $\left(\frac{\partial \tilde{x}_j}{\partial x_i}\right)$ is smooth in x

$\implies (V_\alpha, \varphi_\alpha)$ defines an atlas on TM,

$$\boxed{\dim TM = 2n}$$

Def The <u>tangent bundle</u> of a manifold M is the $2n$-dimensional smooth manifold structure on TM defined by the atlas (V_α, Φ_α) above.

- The projection map
$$p : TM \longrightarrow M, \qquad X_a \in T_a \longmapsto a$$
is <u>smooth</u>, with <u>surjective derivative</u>,
since in loc. coordinates it is given by $p(x_1, \dots, x_n; y_1, \dots, y_n) = (x_1, \dots, x_n)$

- $p^{-1}(a) = T_a$ the fiber of the projection

(Jacobi matrix of p in loc. coord : $\left.\left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{n} & \underbrace{0}_{n} \end{array}\right)\right\}n$)
$\Rightarrow$ rank $= n$

def A <u>vector field</u> on a manifold M is a smooth map $X : M \longrightarrow TM$ such that $p \circ X = id_M$ (– global definition)

Locally; since $p \circ X = id_M$, $X(x_1, \dots, x_n) = (x_1, \dots, x_n; y_1(x), \dots, y_n(x))$
where $y_i(x)$ – smooth functions

I.e., the tangent vector at x:
$$X(x) = \sum_{i=1}^{n} y_i(x) \left(\frac{\partial}{\partial x_i}\right)_x$$
– smoothly varying field of tangent vectors.

Remark More generally, given a projection $p : Q \longrightarrow M$, a <u>"section"</u> of p is a smooth map $s : M \longrightarrow Q$ s.t. $p \circ s = id_M$.

- For $Q = TM$ the tangent bundle, we always have the <u>zero-section</u> – the vector field $X = 0$.

- Multiplying by a bump function μ, we can construct vector fields out of locally defined v.f.s
$$X(x) = \sum \underbrace{y_i(x)}_{\substack{\text{locally-defined} \\ C^\infty \text{ functions}}} \left(\frac{\partial}{\partial x_i}\right)_x$$

Remark Can similarly form the <u>cotangent bundle</u> $T^*M = \bigcup_a T_a^*$ using the basis $(dx_1)_x, \dots, (dx_n)_x$ in T_x^* instead of the dual basis in T_x.

$$\lambda_U : U \times \mathbb{R}^n \longrightarrow \bigcup_{x \in U} T_x^*$$
$$(x_1 \dots x_n; z_1 \dots z_n) \longmapsto \sum z_i (dx_i)_x$$

$$\Phi_U = (\varphi_U, id) \circ \lambda_U^{-1} : \underbrace{\bigcup_{x \in U} T_x^*}_{W} \longrightarrow \varphi_U(U) \times \mathbb{R}^n$$

transition map: $\Psi_\beta \Psi_\alpha^{-1} : (x_1, \dots, x_n; z_1, \dots, z_n) \longmapsto (\tilde{x}_1, \dots, \tilde{x}_n; \sum_i \frac{\partial x_i}{\partial \tilde{x}_1} z_i, \dots, \sum_i \frac{\partial x_i}{\partial \tilde{x}_n} z_i)$

$= \sum_{i,j} z_i \frac{\partial x_i}{\partial \tilde{x}_j} (d\tilde{x}_j)_x \Rightarrow \tilde{z}_j = \sum_i z_i \frac{\partial x_i}{\partial \tilde{x}_j}$

Then, the derivative of $f \in C^\infty(M)$ is a map $df : M \longrightarrow T^*M$ satisfying $p \circ df = id_M$. (but not every section of T^*M is a derivative!)

TM and T^*M are examples of <u>vector bundles</u>.

<u>def</u> A real <u>vector bundle</u> of rank m on a manifold M is a manifold E with a smooth projection map $p : E \longrightarrow M$ s.t.

"local trivialization"

- each fiber $p^{-1}(x)$ has the structure of an m-dimensional real vector space.
- M is covered by open sets $\{U_\alpha\}$ (trivializing neighborhoods) equipped with diffeomorphisms $\psi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times \mathbb{R}^m$ s.t. $proj_1 \circ \psi_\alpha = p$ ($\Rightarrow \psi_\alpha : p^{-1}(x) \to \{x\} \times \mathbb{R}^m$) and $proj_2 \circ \psi_\alpha$ is a linear isomorphism from the v.sp. $p^{-1}(x)$, $x \in U_\alpha$ for any, to the v.space $\mathbb{R}^m$.

<consequence of the previous>

- on the intersection $U_\alpha \cap U_\beta$,
$$\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^m \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^m$$
is of the form $(x, v) \longmapsto (x, g_{\alpha\beta}(x)\, v)$

where $g_{\alpha\beta}(x)$ is a smooth function on $U_\alpha \cap U_\beta$ with values in invertible $m \times m$ matrices.

"transition function"

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for TM, g_{uv} is the Jacobian matrix $\left(\frac{\partial \tilde{x}_i}{\partial x_j} \right)$

for T^*M g_{uv} is its inverse transpose $\left(\frac{\partial x_j}{\partial \tilde{x}_i} \right)$

<u>Ex:</u> $E = M \times \mathbb{R}^m$ - "trivial" vector bundle

- Morphism of vector bundles $\begin{array}{c} E \\ p\downarrow \\ M \end{array}$, $\begin{array}{c} E' \\ \downarrow p' \\ N \end{array}$ is a pair of maps $f : M \to N$, $F : E \to E'$ s.t.

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ p \downarrow & & \downarrow p' \\ M & \xrightarrow{f} & N \end{array}$$

(i.e. $F : p^{-1}(x) \longrightarrow p'^{-1}(f(x))$) and $\forall x \in M$ F gives a <u>linear map</u> $F|_{p^{-1}(x)} : p^{-1}(x) \longrightarrow p'^{-1}(f(x))$.

Vector fields as derivations

X - vector field is a mapping $X: C^\infty(M) \longrightarrow C^\infty(M)$
$\rightarrow X_x \in T_x^{Alg}\ \forall x \in M$

$$f \longmapsto (x \mapsto X_x f) =: X(f)$$

locally: $X(f)(x) = \underbrace{\sum_i y_i(x)\left(\frac{\partial}{\partial x_i}\right)_x}^{X}(f) = \sum_i y_i(x)\frac{\partial f}{\partial x_i}(x)$

- smooth; Leibnitz property: $\boxed{X(fg) = f \cdot X(g) + g \cdot X(f)}$. $(*)$

Linear transformations $X: C^\infty(M) \to C^\infty(M)$ satisfying $(*)$ are called <u>derivations</u> of the ring $C^\infty(M)$.

* Derivations of $C^\infty(M)$ = vector fields on M.

<u>Proposition</u>: Let $X: C^\infty(M) \to C^\infty(M)$ be a lin. map which satisfies $(*)$. Then X is a vector field.

<u>Proof</u>: $\forall a \in M$, $X_a(f) = X(f)(a)$ satisfies the conditions of a tangent vector at a. So, X defines a map $X: M \to TM$ with $p \circ X = id_M$. So, locally it can be written as $X_x = \sum_i y_i(x)\left(\frac{\partial}{\partial x_i}\right)_x$. We need to check that $y_i(x)$ are smooth.

$X(x_i \mu) = y_i(x)$ near a. Since X is $C^\infty \to C^\infty$, $y_i(x)$ is C^∞.

μ: bump function in the coord. nbhd, $=1$ around a

$\square$

• <u>Lie bracket of vector fields</u>

Let X, Y two vector fields on M. We can compose them as operators $C^\infty \to C^\infty$.

$$XY(fg) = X(f\,Y(g) + g\,Y(f)) = X(f)Y(g) + f\,XY(g) + X(g)\,Y(f) + g\,XY(f)$$

$$YX(fg) = Y(f\,X(g) + g\,X(f)) = Y(f)X(g) + f\,YX(g) + Y(g)\,X(f) + g\,YX(f)$$

$\Rightarrow [X,Y] = XY - YX$ satisfies

$$[X,Y](fg) = f \cdot [X,Y](g) + g\,[X,Y](f)$$

$\Rightarrow [X,Y]$ is a vector field.

def The Lie bracket of two vector fields X, Y is the vector field $[X, Y]$.

Ex: $M = \mathbb{R}$, $X = f \frac{d}{dx}$, $Y = g \frac{d}{dx}$ $\Rightarrow [X, Y] = (f g' - g f') \frac{d}{dx}$