We say that $F: M\times[a,b]\to N$ is smooth if it is a restriction of a smooth $\tilde{F}: M\times(a-\varepsilon, b+\varepsilon)\to N$ for some $\varepsilon>0$

⟨ or, more generally: $\tilde{F}: U \to N$, $U \supset M\times[a,b]$, $U \overset{\text{open}}{\subset} M\times\mathbb{R}$ ⟩

Theorem Let $F: M\times[0,1]\to N$ be a smooth map. Set $F_t(x) = F(x,t)$ and consider the induced map on de Rham cohomology $F_t^*: H^p(N)\to H^p(M)$.

Then: $F_1^* = F_0^*$.

Proof Let $a = [\alpha] \in H^p(N)$ ← α: closed form on N

$F^*\alpha \in \Omega^p(M\times[0,1])$ (in fact on $M\times(-\varepsilon, 1+\varepsilon)$)

(#) $F^*\alpha = \beta + dt\wedge\gamma$ ← splitting

β: p-form on M depending on t; γ: $(p-1)$-form on M depending on t

splitting (#) is clear in a coord. system.

more invariantly: $\beta = F_t^*\alpha$.

To get γ: let $\varphi_t : (x,s)\mapsto(x,s+t)$ be the flow on $M\times(a,b)$ – it generates a vector field $X = \frac{\partial}{\partial t}$

then: $\gamma = \iota_X F^*\alpha$

α closed $\Rightarrow$ $F^*\alpha$ closed $\Rightarrow$ $0 = d(\beta + dt \wedge \gamma)$

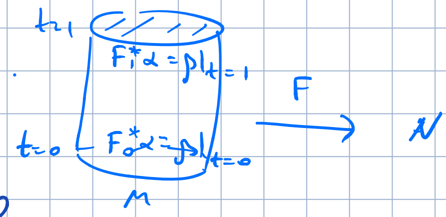
$$= d_M \beta + dt \wedge \frac{\partial \beta}{\partial t} - dt \wedge d_M \gamma$$

d_M = exterior derivative in the variables of M.

$$\Rightarrow \frac{\partial \beta}{\partial t} = d_M \gamma \quad \Rightarrow F_1^*\alpha - F_0^*\alpha$$

$$\frac{\partial}{\partial t} F_t^* \alpha$$

$$= \int_0^1 dt \underbrace{\frac{\partial}{\partial t} F_t^*\alpha}_{\frac{\partial \beta}{\partial t} = d_M \gamma} = d_M \int_0^1 dt\, \gamma$$



$\Rightarrow$ closed forms $F_1^*\alpha$, $F_0^*\alpha$ differ by an exact form

$\Rightarrow$ $F_1^* a = F_0^* a$

$\blacksquare$

<u>Corollary</u>: The de Rham cohomology groups of $M = \mathbb{R}^n$ are zero for $p > 0$.

<u>Proof</u>: set $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$
$(x, t) \mapsto tx$

$F_1 = id$

F_0 maps $\mathbb{R}^n$ to $0 \in \mathbb{R}^n$.

- constant map

$$\underset{=id}{F_1^*} : H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n)$$

$\Rightarrow$ DF_0 vanishes $\Rightarrow$

for any $\alpha \in \Omega^{>0}(\mathbb{R}^n)$, $\boxed{F_0^*\alpha = 0}$

$$\Rightarrow F_0^* : H^p(\mathbb{R}^n) \xrightarrow[\text{zero map}]{} H^p(\mathbb{R}^n), \quad p > 0$$

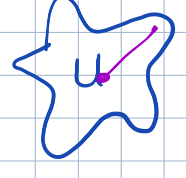
by Proposition, $F_1^* = F_0^* \Rightarrow H^p(\mathbb{R}^n) = 0$, $p > 0$

$\blacksquare$

- $\mathbb{R}^n$ is connected $\Rightarrow$ $H^0(\mathbb{R}^n) = \{$locally constant functions on $\mathbb{R}^n\}$ $\cong \mathbb{R}$

- By a similar argument:

- <u>Poincaré lemma</u>: if $U \underset{\text{open}}{\subset} \mathbb{R}^n$ a "star-shaped region"
$\forall x \in U$, $t \in [0,1]$, $tx \in U$

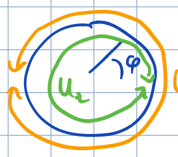


then $H^p(U) = \begin{cases} \mathbb{R} & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$

- $H^p(M \times \mathbb{R}^n) \cong H^p(M)$ for any manifold M

$F_t(a, x) = (a, tx)$ $\sim$ deformation retraction of $M \times \mathbb{R}^n$ onto M

* **Example:** $H^1(S^1)$



$S^1 = \{ e^{i\varphi} \in \mathbb{C} \}$
parameterize by the angle $\varphi \in \mathbb{R}/_{2\pi}\mathbb{Z}$

$\mu = d\varphi$ is a ~~nowhere-vanishing~~ 1-form on S^1.
μ defined by this f-la in $U_1 = S^1 \setminus \{-1\}$ where $\varphi \in (-\pi, \pi)$
and $U_2 = S^1 \setminus \{+1\}$ where $\varphi \in (0, 2\pi)$

$d\mu = 0$ obviously (e.g. for degree reason)
$\Rightarrow$ μ closed

assume $\mu = d\underbrace{f}_{\text{global function on } S^1}$ S^1 compact $\Rightarrow$ f must have a min and a max
$\Rightarrow$ df must vanish somewhere
but μ is nonvanishing! $\Rightarrow$ μ is not exact.

So: $H^1(S^1) \neq 0$
and contains the nonzero class $[\mu]$

Let $\alpha \in \Omega^1(S^1)$ any form
$\alpha = \underbrace{g(\varphi)}_{\text{periodic function of } \varphi} d\varphi$

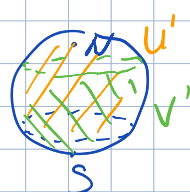
want $\alpha = dh$ $\Rightarrow$ $g(\varphi) = h'(\varphi)$
solution $h(\varphi) = \int_0^\varphi ds\, g(s)$ $(+C)$
this solution is periodic if $\int_0^{2\pi} ds\, g(s) = 0$

generally: $g(\varphi) = \underbrace{\frac{1}{2\pi} \int_0^{2\pi} ds\, g(s)}_{g_0 \text{ - mean value of } g \text{ on } S^1} + \tilde{g}(\varphi)$ with $\tilde{g}(\varphi) = h'(\varphi)$, $\tilde{g}(\varphi) = g(\varphi) - g_0$, $h(\varphi) = \int_0^\varphi ds\, \tilde{g}(s)$

So: $\alpha = g_0 \underbrace{\mu}_{d\varphi} + \underbrace{dh}_{\text{exact form}}$ $\Rightarrow$ $H^1(S^1) = \mathrm{Span}[\mu] = \mathbb{R}$. ✓

Theorem: For $n > 0$, $H^p(S^n) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } n \\ 0 & \text{otherwise} \end{cases}$

Proof Let $n > 1$ (case $n = 1$ was discussed above). Let $1 < p < n$.
Let $\alpha \in \Omega^p_{closed}(S^n)$



$U = S^n \setminus S \xrightarrow{\varphi_U} \mathbb{R}^n$
$V = S^n \setminus N \xrightarrow{\varphi_V} \mathbb{R}^n$
stereographic charts on S^n

$\alpha|_U = du$ $\leftarrow$ since $H^p(\mathbb{R}^n) = 0$)
$\alpha|_V = dv$ for some $u \in \Omega^{p-1}(U)$, $v \in \Omega^{p-1}(V)$

On $U \cap V$: $0 = \alpha|_{U\cap V} - \alpha|_{U \cap V} = du - dv = d(u - v)$

$$\Rightarrow \quad u - v \in \Omega^{p-1}_{closed}(\underbrace{U\cap V}_{\underset{diffeo}{\cong}\ \mathbb{R}\times S^{n-1}})$$

$$H^{p-1}(\mathbb{R}\times S^{n-1}) = H^{p-1}(S^{n-1}) \underset{\text{by induction}}{=} 0 \quad \text{for} \quad \underline{\underline{1<p<n}}$$

$$\Rightarrow u - v = d\omega \,, \quad \omega \in \Omega^{p-2}(U\cap V)$$

Let $U' = \varphi_U^{-1}(B_2(0))$
$V' = \varphi_V^{-1}(B_2(0))$

Let ψ = bump function with support on $U\cap V$ s.t. $\psi = 1$ on $U'\cap V'$

$\psi\cdot\omega$ - global $(p-2)$-form on S^n (extended by zero)

define $\beta = \begin{cases} u & \text{on } U' \\ v + d(\psi\omega) & \text{on } V'\end{cases}$ - global $(p-1)$-form (restrictions to $U'\cap V'$ agree)

then $d\beta = \alpha \Rightarrow \alpha$ is exact! $\Rightarrow H^p(S^n) = 0$ for $1<p<n$

- If $p=1$, $u - v \in \Omega^0_{closed}(U\cap V)$
  $= c$ - a constant function (using that $U\cap V$ is connected, for $n>1$)

  $\Rightarrow \beta = \begin{cases} u & \text{on } U\\ v+c & \text{on } V\end{cases}$ - global function, $d\beta = \alpha$ (agree on overlap)

- If $p=n$, $u-v$ defines a class in $H^{p-1}(U\cap V) \cong H^{p-1}(S^{n-1}) \underset{induction}{\cong} \mathbb{R}$

  Let $H^{p-1}(S^{n-1}) = \mathrm{Span}[\omega]$, $\omega \in \Omega^{p-1}_{closed}(S^{n-1})$

  $$U\cap V \xrightarrow[\cong]{\Phi} S^{n-1}\times\mathbb{R} \xrightarrow{\pi} S^{n-1}$$

  So: $u - v = \lambda\Phi^*\omega + dw$, some $\lambda\in\mathbb{R}$. If $\lambda=0$, then we do as above and find a global $(p-1)$ form β s.t. $\alpha = d\beta$.

- λ is linear in α
  and independent of the choice of u, v (shifting them by an exact term can be absorbed into w)

$\Rightarrow \dim H^n(S^n) \leq 1$

- Need to find α with nonzero λ. Set $\alpha = \Phi^*(\psi(t)\,dt\wedge\omega) \in \Omega^n(S^n)$

  ψ - bump function on $\mathbb{R}$

  extended by zero outside $\mathrm{supp}\,\Phi^*\psi$

$$u = \Phi^*\left(\left(\int_{-\infty}^t \psi(s)\,ds\right)\omega\right) \in \Omega^{p-1}(U)$$

extension by zero from $U\cap V$ to U

s.t. $du = \alpha$

$$v = \Phi^*\left(-\int_t^\infty \psi(s)\,ds\,\omega\right) \in \Omega^{p-1}(V)$$

$$u - v = \underbrace{\left(\int_{-\infty}^\infty\psi(s)\,ds\right)}_{\lambda>0}\Phi^*\omega \quad \text{on } U\cap V$$

$\square$

Integration of forms

Orientation Recall a change of variables formula in a multiple integral

$$\int f(y_1,\dots,y_n)\,dy_1\dots dy_n = \int f(y_1(x)\cdots y_n(x))\left|\det \frac{\partial y_i}{\partial x_j}\right| dx_1\dots dx_n \qquad (*)$$

compare to: the change of coords for an n-form on an n-mfd:

$$\theta = f(y_1,\dots,y_n)\,dy_1\wedge\dots\wedge dy_n = f(y_1(x),\dots,y_n(x))\left(\sum_{i_1}\frac{\partial y_1}{\partial x_{i_1}}dx_{i_1}\right)\wedge\dots\wedge\left(\sum_{i_n}\frac{\partial y_n}{\partial x_{i_n}}dx_{i_n}\right)$$

$$= f(y_1(x),\dots,y_n(x))\cdot\det\left(\frac{\partial y_i}{\partial x_j}\right)\cdot dx_1\wedge\dots\wedge dx_n$$

So: the difference is the absolute value $|\det|$ in $(*)$. If we can deal with that, we should be able to assign a coord-independent value to $\int_M \underset{\uparrow\,n\text{-form}}{\alpha}$.

... Dealing with $|\det|$ vs $\det$ = orientation ...

def An n-mfd is orientable if it has an everywhere non-vanishing n-form ω.

def Let M be an orientable n-manifold. An orientation on M is an equiv. class of non-vanishing n-forms ω where $\omega\sim\omega'$ if $\omega'=f\omega$ with $f>0$.

- A connected orientable mfd has two orientations $[\pm\omega]$.

Ex 1: $f:\mathbb{R}^{n+1}\to\mathbb{R}$, $\quad M=f^{-1}(c)\subset\mathbb{R}^{n+1}$ submanifold

$c \in$ regular value of f (by regular value levelset thm)

if $\left.\frac{\partial f}{\partial x_i}\right|_a \neq 0$, then $x_1,\dots,\hat{x}_i,\dots,x_{n+1}$ are loc. coords on M near a.

On such a patch, consider $\omega = (-1)^i \frac{1}{\frac{\partial f}{\partial x_i}}\,dx_1\wedge\dots\widehat{dx_i}\dots\wedge dx_{n+1}$ (#) - non-vanishing n-form

if $\left.\frac{\partial f}{\partial x_j}\right|_a \neq 0$ also, then $\omega = (-1)^j \frac{1}{\partial f/\partial x_j}\,dx_1\wedge\dots\widehat{dx_j}\dots\wedge dx_{n+1} \Leftarrow$ substitute $dx_j = -\frac{1}{\frac{\partial f}{\partial x_j}}\left(\frac{\partial f}{\partial x_i}dx_i+\dots\right)$

$\sum_j \frac{\partial f}{\partial x_j}dx_j = 0$ on M

$\Rightarrow$ (#) defines for all charts a non-vanishing n-form

$\Rightarrow$ M is orientable

E.g. $M=S^n$ with $\omega = (-1)^i\frac{1}{x_i}\,dx_1\wedge\dots\widehat{dx_i}\dots\wedge dx_{n+1}$ (@)

Ex 2 Consider $\mathbb{RP}^n$, $p: S^n \longrightarrow \mathbb{RP}^n$
$v \longmapsto \text{Span}\{v\}$
unit vector in $\mathbb{R}^{n+1}$

~~if $x_1 \neq 0$, use $(x_2 \cdots x_{n+1})$ coordinates on S^n, $\left(\frac{x_2}{x_1}, \ldots, \frac{x_{n+1}}{x_1}\right)$ coords. on $\mathbb{RP}^n$~~
~~$p(x) = \frac{x}{\sqrt{1-\|x\|^2}}$ smooth with smooth inverse $q(y) = \frac{y}{\sqrt{1+\|y\|^2}}$~~

Let $\sigma: S^n \to S^n$ be the diffeo $\sigma(x) = -x$. Then

$$\underset{(@)}{\sigma^* \omega} = (-1)^i \frac{1}{-x_i} d(-x_1) \wedge \cdots \widehat{d(-x_i)} \cdots \wedge d(-x_{n+1}) = (-1)^{n+1} \omega$$

Suppose $\mathbb{RP}^n$ is orientable. Then it has a nonvanishing n-form Θ $\Rightarrow$ $p^*\Theta$ is a nonvanishing n-form on S^n
and so $p^*\Theta = f\omega$,
for f a nonvanishing function.

$p \circ \sigma = p \Rightarrow f\omega = p^*\Theta = \sigma^* p^* \Theta = \sigma^*(f\omega)$
$= (\sigma^* f) \cdot (-1)^{n+1} \omega$

Thus, if $n = 2m$ even, then $f = -\sigma^* f$, i.e. $f(a) = -f(-a)$

So, if $f(a) > 0$, then $f(-a) < 0$. But S^n is connected, so this means that f must vanish somewhere - contradiction!

$\Rightarrow$ $\mathbb{RP}^{2m}$ is non-orientable!

Proposition A manifold is orientable iff it has a covering by coord. charts such that $\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0$ on the intersection.

Proof Assume M is orientable, ω a non-vanishing n-form. In a coord. chart, $\omega = f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$. After possibly making a coord. change $x_1 \mapsto c - x_1$, we have coords. s.t. $f > 0$.

Look at two such overlapping charts:

$$\omega = g(y_1, \ldots, y_n)\, dy_1 \wedge \cdots \wedge dy_n = g(y_1(x), \ldots, y_n(x)) \left(\det \frac{\partial y_i}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx_n$$
$$= f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$$

since $f > 0$, $g > 0$, we have $\det > 0$.

Conversely: suppose we have such coords. on $\{U_\alpha\}$. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$

Set $\omega = \sum_\alpha \varphi_\alpha \, dy_1^\alpha \wedge \dots \wedge dy_n^\alpha$

On a chart U_β with coords $x_1, \dots, x_n$, we have

$$\omega|_{U_\beta} = \sum_\alpha \underbrace{\varphi_\alpha}_{\geq 0} \underbrace{\det\left(\frac{\partial y_i^\alpha}{\partial x_j}\right)}_{>0} dx_1 \wedge \dots \wedge dx_n$$ - non-vanishing! ☒

· Integration   suppose M is orientable and we have chosen an orientation.
We will define the integral $\boxed{\int_M \theta}$ of any n-form θ of compact support on M.

· choose $\{U_\alpha\}$ - coordinate covering $\theta|_{U_\alpha} = f_\alpha(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n$

let $\{\varphi_i\}$ - partition of unity subordinate to $\{U_\alpha\}$

$$\varphi_i \theta|_{U_\alpha} = \underbrace{g_i(x_1, \dots, x_n)}_{\text{smooth function with compact support on the entire } \mathbb{R}^n} dx_1 \wedge \dots \wedge dx_n$$

$\operatorname{supp} \varphi_i \subset U_\alpha$

Define $$\int_M \theta := \left(\sum_i \int_M \varphi_i \theta :=\right) \sum_i \int_{\mathbb{R}^n} g_i(x_1, \dots, x_n)\, dx_1 \cdots dx_n$$

- $\operatorname{supp} \theta$ compact $\Rightarrow \varphi_i \theta \neq 0$ for finitely many i's $\Rightarrow$ finitely many terms are nonzero
+ $\{\operatorname{supp} \varphi_\alpha\}$ are locally finite   (finitely many $\operatorname{supp} \varphi_i$'s intersect $\operatorname{supp} \theta$)

$\int_M \theta$ is well-defined because of the change of variables f-la for integration
+ consistent choice of sign of det Jac from orientation.