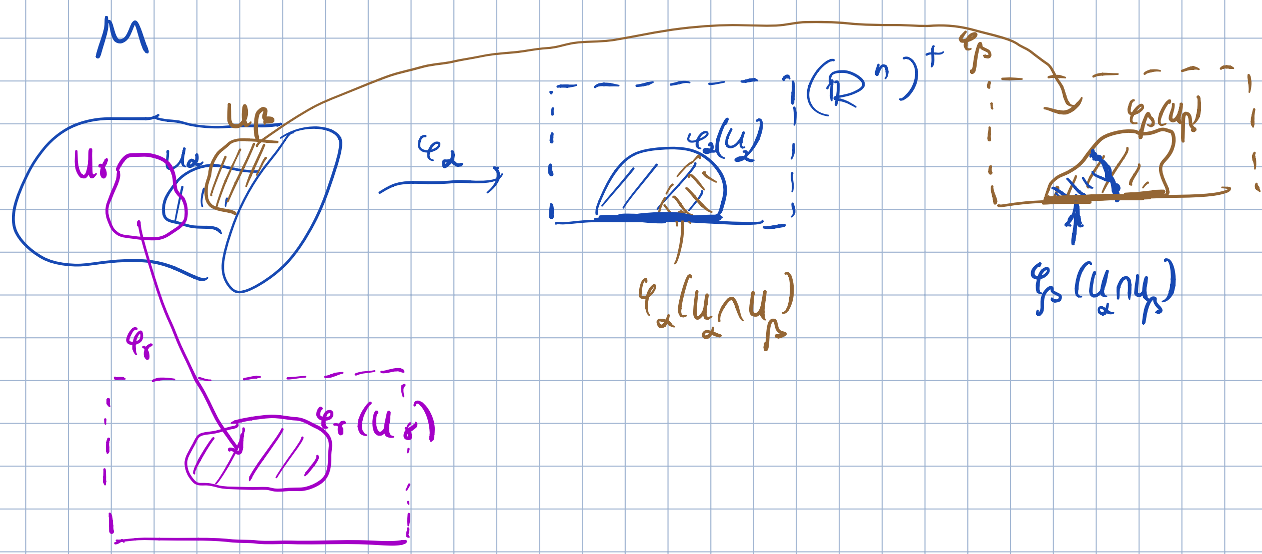


LAST TIME:

- <u>Orientation</u> on an n-manifold M = a nowhere-vanishing n-form ω on M up to equivalence $\omega \sim f\omega$, $f \in C^\infty(M)$, $f > 0$

- S^n is orientable
  $\mathbb{RP}^{2m}$ is non-orientable

<u>Proposition</u> A mfd is orientable iff it has a covering by coord. charts s.t. $\det \dfrac{\partial y_i}{\partial x_j} > 0$ on every overlap.

<u>Proof</u> Assume M is orientable ω - nonvanishing n-form in a coord. chart on M, $\boxed{\omega = f(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n}$ After possibly making a coord. change $x_1 \longmapsto -x_1$ we have coords s.t. $f > 0$

On an overlap

$$\omega = \underbrace{g(y_1 \dots y_n)}_{>0}\, dy_1 \wedge \dots \wedge dy_n = \underbrace{g(y_1(x), \dots, y_n(x))}_{>0} \left(\det \frac{\partial y_i}{\partial x_j}\right) dx_1 \wedge \dots \wedge dx_n$$

$$= \underbrace{f(x_1, \dots, x_n)}_{>0} \quad dx_1 \wedge \dots \wedge dx_n$$

$$\Rightarrow \boxed{\det \frac{\partial y_i}{\partial x_j} > 0}$$

Conversely: suppose we have such coords on $\{U_\alpha\}$. Let $\{\varphi_\alpha\}$ - part. of unity subordinate to $\{U_\alpha\}$

$$\omega = \sum_\alpha \varphi_\alpha \underbrace{dy_1^\alpha \wedge \dots dy_n^\alpha} \in \Omega^n(M)$$

On a chart U_β with coords $x_1 \dots x_n$

$$\omega|_{U_\beta} = \underbrace{\Big(\sum_\alpha \underset{\geq 0}{\varphi_\alpha} \underbrace{\det \frac{\partial y_i^\alpha}{\partial x_j}}_{>0}\Big)}_{>0} dx_1 \wedge \dots \wedge dx_n \quad \text{- non-vanishing.}$$

$\square$

Integration

Suppose M is an orientable n-mfd and we have chosen an orientation.

Fix Θ an n-form on M with compact support. $\Theta \in \Omega^n_c(M)$ (c: compact support)

want to define $\int_M \Theta$.

Choose $\{U_\alpha\}$ covering by coord. charts compatible with the orientation $\quad \Theta|_{U_\alpha} = f(x_1, \dots, x_n) dx_1 \wedge \dots dx_n$

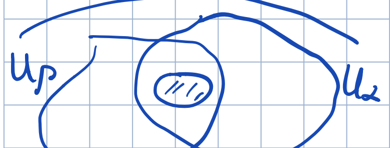
$\{\varphi_i\}$ - part of unity $\quad \varphi_i \Theta|_{U_\alpha} = \underbrace{g_i(x_1, \dots, x_n)} dx_1 \wedge \dots \wedge dx_n$

supp $\varphi_i \subset U_\alpha \quad$ sm. function with cpt support on $\mathbb{R}^n$

$$\boxed{\int_M \Theta = \sum_i \int_M \varphi_i \Theta = \sum_i \int_{\mathbb{R}^n} g_i(x_1 \dots x_n)\, dx_1 \dots dx_n}$$

$\boxed{\text{supp } \Theta \text{ cpt, } \{\text{supp } \varphi_i\} \text{ are loc. finite}} \Rightarrow \underline{\varphi_i \Theta \neq 0 \text{ for finitely many } i\text{'s}} \Rightarrow$ finitely many in $\sum_i$ are $\neq 0$.

$\int_M \Theta$ is well-defined because of change of variables f-la for integral + consistent choice of sign of det Jac from orientation



Guillemin-Pollack, p. 167-168

Properties: ① $\int_M : \Omega^n_c(M) \to \mathbb{R}$ is a linear map

$$\int_M (\alpha+\beta) = \int_M \alpha + \int_M \beta \qquad \int_M c\alpha = c\int_M \alpha$$

② changing the orientation on M results in changing the sign of $\int_M \theta$.
(say, M is connected)

③ if $F: M \to N$ is orientation-preserving diffeomorphism and $\theta \in \Omega^n_c(N)$

$$\boxed{\int_N \theta = \int_M F^*\theta}$$

Stokes' Theorem

simple version

Theorem: Let M an oriented n-manifold and $\alpha \in \Omega^{n-1}_c(M)$. Then

$$\boxed{\int_M d\alpha = 0}$$

Proof Choose a part. of unity $\{\varphi_i\}$ subordinate to a coord. cover $\{U_\alpha\}$

$\alpha = \sum_i \varphi_i \alpha$ In a chart:

$$\varphi_i \alpha = \underline{a_1}\, dx_2 \wedge \dots \wedge dx_n - \underline{a_2}\, dx_1 \wedge dx_3 \wedge \dots \wedge dx_n + \dots$$

$$d(\varphi_i \alpha) = \left(\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \dots + \frac{\partial a_n}{\partial x_n}\right) dx_1 \wedge \dots \wedge dx_n$$

$$\Rightarrow \int_M \varphi_i \alpha = \int_{U_\alpha} \varphi_i \alpha = \int_{\mathbb{R}^n} \left(\underline{\underline{\frac{\partial a_1}{\partial x_1}}} + \dots + \frac{\partial a_n}{\partial x_n}\right) dx_1 \dots dx_n = 0$$

Consider $\displaystyle\int_{\mathbb{R}^n} \frac{\partial a_1}{\partial x_1} dx_1 \dots dx_n \underset{\text{Fubini}}{=} \int_{\mathbb{R}} dx_n \int_{\mathbb{R}} dx_{n-1} \dots \underbrace{\int_{\mathbb{R}} dx_1 \frac{\partial a_1}{\partial x_1}}_{\lim_{N\to\infty} a_1\big|_{-N}^{N}} = 0$

fund. thm of calc.

$= 0$ since a_1 has cpt support

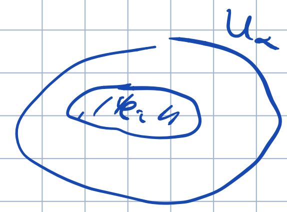
other terms vanish similarly

$\square$

Proposition Let M be a cpt orientable n-mfd.
Then $H^n(M) \neq 0$.

Proof: M orientable $\Rightarrow$ ω - nonvanishing n-form $\qquad \omega$ is closed

$[\omega] \in H^n(M)$



choose the orientation determined by ω

$$\int_M \omega = \sum_i \int \underbrace{f_i}_{\geq 0} \, dx_1 \cdots dx_n > 0 \qquad \varphi_i \omega\big|_{U_\alpha}$$

each f_i is > 0 somewhere.

assume $\omega = d\alpha \Rightarrow \int_M \omega = \int_M d\alpha = 0 \Rightarrow \omega$ cannot be exact

$\Rightarrow [\omega] \neq 0$ in $H^n(M)$ $\square$