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# MATH 60330 "Basic Geometry and Topology"

MWF HH117, $11^{40} - 12^{30}$

**Organizational:**

Midterm: TBA ~ Sep 23 (take-home)

Final: week of Nov 16 (?)

Homeworks: first batch assigned on Fri 8/14, due Fri 8/21 via Sakai/Assignments

Quizzes: every 1-2 week, 10 min in the beginning of the class

Office hours: Wed, 5-6 pm via Zoom (link on web page)

Grading:

| midterm | | final | | quizzes | | homework |
|---|---|---|---|---|---|---|
| 100 | + | 150 | + | 100 | + | 100 |

- report your seat location using here.nd.edu/seat (Aug 10 - 19, each time class meets)
  10, 12, 13, 17, 19

Links

Course web page: www3.nd.edu/~pmnev/f20/MATH60330.html

Syllabus: www3.nd.edu/~pmnev/f20/syllabus.pdf

Zoom (office hours, classes):

ID: 956 2604 7322

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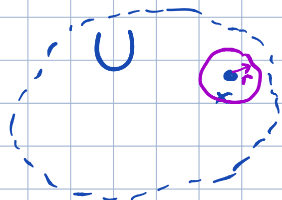
8/10/2020



# Point set topology

## Open subsets of $\mathbb{R}^n$

**def** A subset $U \subset \mathbb{R}^n$ is *open* if for each point $x \in U$, there is some $r > 0$ s.t. $\underbrace{B_r(x)}_{\{y \in \mathbb{R}^n \mid dist(x,y) < r\} \text{ - open ball}} \subset U$

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Equivalently: $U \subset \mathbb{R}^n$ is open iff $U$ is a union of open balls

**def** A map $f: \mathbb{R}^m \to \mathbb{R}^n$ is *continuous* if $\forall$ open $U \subset \mathbb{R}^n$, $f^{-1}(U)$ is open in $\mathbb{R}^m$. (equivalent to $\varepsilon$-$\delta$ definition!)

- More generally: $V \xrightarrow{f} W$ is continuous if $\forall$ open $U \subset \mathbb{R}^n$, $f^{-1}(U) \subset \mathbb{R}^m$ is open

$V \cap$ open $\mathbb{R}^m$, $W \cap$ open $\mathbb{R}^n$

**Ex:**

1. (from calculus) $V \xrightarrow{f} \mathbb{R}$, $V \subset \mathbb{R}$, is continuous for $f$: - polynomials, rational functions, exp, trig functions

   $V$ = domain of $f$.

2. maps $\mathbb{R}^2 \to \mathbb{R}$
   $(x_1, x_2) \longmapsto x_1 + x_2$
   and $(x_1, x_2) \longmapsto x_1 x_2$

3. projection maps $p_k : \mathbb{R}^m \to \mathbb{R}$
   $(x_1, \dots, x_m) \longmapsto x_k$

**Lemma:** a composition of continuous maps is continuous
< Proof: exercise >

**Lemma*:** $f: V \to \mathbb{R}^n$, $x \longmapsto f(x) = (f_1(x), \dots, f_n(x))$ is cont. iff all component maps $f_k : V \to \mathbb{R}$ are cont.

<proof: later>

**Lemma:** If $f_1, f_2 : \underset{\mathbb{R}}{\overset{}{V}} \to \mathbb{R}$ ($V \subset \mathbb{R}$) cont. maps, then $f_1 + f_2$ and $f_1 \cdot f_2$ are cont.

**Proof:**
$$V \xrightarrow{f=(f_1,f_2)} \mathbb{R}^2 \xrightarrow{+} \mathbb{R}$$
$$x \longmapsto (f_1(x), f_2(x)) \longmapsto f_1(x)+f_2(x)$$
i.e. $f_1+f_2$ is cont. as a composition of cont. maps.

similarly for $f_1 \cdot f_2$

**Ex:** • $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, $f(x) = \sum_{i_1,\dots,i_n} \underbrace{a_{i_1 \dots i_n}}_{\text{coefficients}} x_1^{i_1} \cdots x_n^{i_n}$ – polynomial

each summand is a product of a const map $\mathbb{R}^n \to \mathbb{R}$ and proj. maps $p_k : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto x_k$ $\Rightarrow$ is cont. $\Rightarrow$ the whole sum is continuous

• $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ – vect. space of $n\times n$ matrices.

matrix multiplication
$$M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$$
$$(A, B) \longmapsto AB$$
is a cont. map

– by Lm *, since each matrix entry of $AB$ is a polynomial and hence cont. fun. of entries of $A, B$.

## Topological spaces

**def** A *topological space* is a set $X$ together with a collection $\mathcal{T}$ of subsets of $X$ ("open subsets") s.t.

(i) $X$ and $\emptyset$ are open

(ii) any union of open sets is open

(iii) intersection of any *finite* number of open sets is open.

• a complement of an open set in $X$ is a "closed set".

**def** a map $f: X \to Y$ between top spaces is *continuous* if $f^{-1}(V)$ is open in $X$ for any open $V \subset Y$.

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Examples of top. spaces: 1. $\mathcal{T}$ = open subsets of $\mathbb{R}^n$ (unions of open balls).
$\mathcal{T}$ – the "standard topology" on $\mathbb{R}^n$ or "metric topology" on $\mathbb{R}^n$

2. $X$ - set, $\mathcal{T} = \{\text{all subsets of } X\}$ - "discrete topology"

Rem: any map $\underset{\text{discr.}}{X} \longrightarrow Y$ is continuous!

$\mathbb{R}^n \longrightarrow \underset{\text{discr.}}{X}$ is cont. iff it is a constant map!

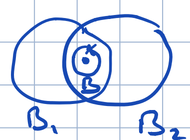
3. $X$ set, $\mathcal{T} = \{\emptyset, X\}$ - "indiscrete topology"

"Basis" for a topology. Ex: opens in $\mathbb{R}^n$ are unions of "standard opens", $B_r(x)$

Lemma: Let $\mathcal{B}$ be a collection of subsets of a set $X$ satisfying

(a) every point $x \in X$ belongs to some $B \in \mathcal{B}$

(b) if $B_1, B_2 \in \mathcal{B}$ then $\forall x \in B_1 \cap B_2$ $\exists$ $\underset{\underset{x}{\cup}}{B} \in \mathcal{B}$ with $B \subset B_1 \cap B_2$

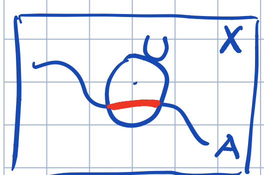



Then $\mathcal{T} := \{\text{unions of subsets belonging to } \mathcal{B}\}$ is a topology on $X$.

def If (a), (b) are satisfied, $\mathcal{B}$ is called a basis for the topology $\mathcal{T}$.
(Or we say: $\mathcal{B}$ generates the topology $\mathcal{T}$)

Subspace topology

def Let $X$ be a top space, $A \subset X$ subset. Then
$\mathcal{T} = \{A \cap U \mid \underset{\text{open}}{U \subset X}\}$ is a topology on $A$ called the subspace topology.




Examples (subspaces of $\mathbb{R}^n$)

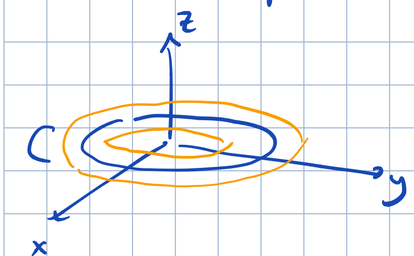
1. n-disk $D^n := \{x \in \mathbb{R}^n \mid \|x\| \le 1\} \subset \mathbb{R}^n$, $D^n_r = \{x \mid \|x\| < r\}$ $r > 0$ radius

2. n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} \subset \mathbb{R}^{n+1}$

3. torus $T = \{v \in \mathbb{R}^3 \mid \mathrm{dist}(v, C) = r\}$, $0 < r < 1$,

here $C = \{(x, y, 0) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^3$;
unit circle in xy plane

$\mathrm{dist}(v, C) = \inf_{w \in C} \mathrm{dist}(v, w)$




4. $GL_n(\mathbb{R}) = \{\text{vector space isomorphisms } f: \mathbb{R}^n \to \mathbb{R}^n\}$
$\longleftrightarrow \{\text{invertible } n\times n \text{ matrices}\} \subset M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$
$\det \neq 0$

groups

5. $SL_n(\mathbb{R}) = \{A \in M_{n\times n} \mid \det A = 1\} \quad \subset M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$

6. $O_n(\mathbb{R}) = \{A \mid A^T A = 1\} \quad \subset M_{n\times n}(\mathbb{R})$

$=$ linear isometries $f: \mathbb{R}^n \to \mathbb{R}^n$

7. $SO_n(\mathbb{R}) = \{A \in O_n(\mathbb{R}) \mid \det A = 1\} \quad \subset M_{n\times n}(\mathbb{R})$

8. <u>Stiefel manifold</u> $V_k(\mathbb{R}^n) = \{(v_1, \dots, v_k) \mid v_i \in \mathbb{R}^n,\ v_i\text{'s are orthonormal}\}$

i.e. $v_i \cdot v_j = \delta_{ij}$

$\subset M_{n\times k}(\mathbb{R}) = \mathbb{R}^{nk}$

~~Stopped here~~

<u>Lemma</u>** (continuity criterion for maps to a subspace)
Let $X, Y$ top. spaces, $\underset{\text{subset}}{A \subset Y}$ with subspace topology. Then

(a) The inclusion map $i: A \to Y$ is continuous

(b) $f: X \to A$ is cont. iff the composition $X \xrightarrow{f} A \xrightarrow{i} Y$ is cont.

<u>Ex</u> (cont. maps involving subspaces)

1. $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$
$\quad A \mapsto A^{-1}$

by Lm** (suffices to prove continuity of $GL_n \xrightarrow{A^{-1}} GL_n \xhookrightarrow{i} \mathrm{Mat}_{n\times n}$ - by Lm* suffices to check continuity component-wise)

2. Let $G$ be one of $SL_n(\mathbb{R}), O_n(\mathbb{R}), SO_n(\mathbb{R})$ with subspace topology as subsets of $M_{n\times n}(\mathbb{R})$.
Then the map $G \to G$, $A \mapsto A^{-1}$ is continuous.

- follows from 1. and:

<u>Lemma</u> If
$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, hook, "i"] & B \arrow[u, hook, "j"'] \end{tikzcd}$$
and $f(A) \subset B$, then $f|_A : A \to B$ is continuous w.r.t. subspace topology on $A, B$.

<u>Proof</u>:
$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, hook, "i"] \arrow[r, "f|_A"'] & B \arrow[u, hook, "j"'] \end{tikzcd}$$

$i, j$ cont. $(\text{Lm}^{**}(a)) \Rightarrow f \circ i$ - cont
$\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{j \circ f|_A}$ - cont $\Rightarrow$

$\overset{\Rightarrow}{\scriptstyle \text{Lm}^{**}(b)}\ f|_A$ - cont.