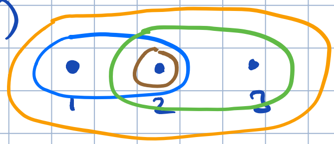
LAST TIME:

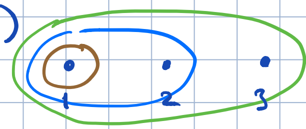
- $f: X \longrightarrow Y$ cont. bijection is a homeomorphism if X is compact, Y is Hausdorff.

- Heine-Borel: $K \subset \mathbb{R}^n$ is compact iff K is closed and bounded.

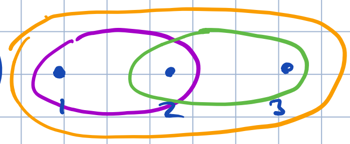
Quiz I, 8/19/20

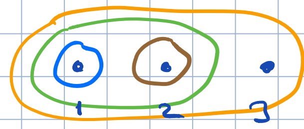
1. Which is NOT a topology on $\{1,2,3\}$?

A)


B)


+ $\emptyset$

C)


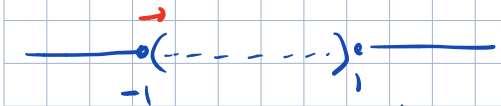
D)


2. Which among those in 1 which are topologies are Hausdorff ?

3. What is a <u>compact</u> topological space? (Definition)

A quotient of a Hausdorff space which is not Hausdorff

$$X = \mathbb{R}/(-1,1)$$



points -1, 1 in X do not have disj. open neighborhoods

U, V disjoint - open in X $\rightsquigarrow$ $p^{-1}(U)$, $p^{-1}(V)$ disjoint, open in $\mathbb{R}$

$U \ni -1$, $V \ni +1$; $p^{-1}(U) \ni -1$, $p^{-1}(V) \ni 1$

$p^{-1}(U)$ overlaps with $(-1,1)$ - must contain $(-1,1)$

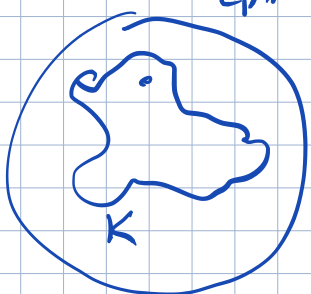
$p^{-1}(V)$ ——— " ——— ——— " ———

$\Rightarrow$ cannot be disjoint $\to$ contradiction !

Heine-Borel theorem

- fact 1: $[a,b]$ a closed interval in $\mathbb{R}$ is compact
- fact 2: if $X_1, \dots, X_n$ are cpt top. spaces then $X_1 \times \dots \times X_n$ is cpt.

- Let $K \subset \mathbb{R}^n$ cpt $\Rightarrow K$ closed (Lemma γ)



$\{B_r(0) \cap K\}_{r>0}$ - open cover of K

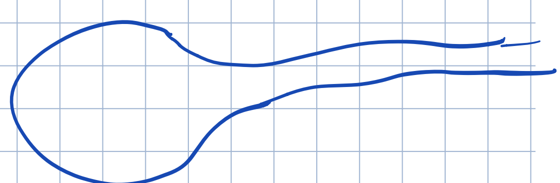
finite subcover $\{B_{r_i}(0) \cap K\}$

R - largest r_i - $B_R(0) \supset K$

K bounded.

- $\Leftarrow$ K closed bounded $\Rightarrow$ $K \subset B_R(0) \subset [-R,R]^{\times n}$, K closed, $[-R,R]^{\times n}$ cpt $\leftarrow$ using facts 1, 2

$\Rightarrow$ K is compact ! (Lemma β) $\boxed{}$



Corollary of Heine-Borel

if $f: X \to \mathbb{R}$ cont., X cpt, then f has a minimum and a maximum.

Proof: $K = f(X) \subset \mathbb{R}$ $\Rightarrow$ K is bounded cpt (Lemma α)

$\Rightarrow K$ has a supremum b $\quad (b-\varepsilon, b]$ $\quad$ $\sup = b$ - is a limit of a sequence in K

and an infimum a

$\Rightarrow b \in K$ $\quad$ b = maximum

since K is closed

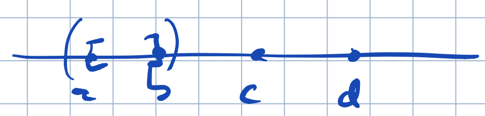
$\Rightarrow \exists\, x_{max}$ s.t. $f(x_{max}) = b$

$x_{max} \in X$ $\quad$ $f(x) \leq f(x_{max})$ $\boxed{\checkmark}$

Connected space

def A top. space X is called connected if it can't be written as $X = U \amalg V$ with U, V non-empty, disjoint open subsets of X.

Ex Let



$a < b < c < d$

- $X = (a, b) \cup (c, d)$ is not connected
- $X = [a, b] \cup [c, d]$ — " —

open set in X $\quad$ open set in X

Lemma: Any interval $I \subset \mathbb{R}$ (open, closed, half-open, bounded or not)

Proof: assume $I = U \amalg V$ $\quad$ assume $u < v$

$u \in U$, $v \in V$

$[u, v] = U' \amalg V'$

$U' = U \cap [u, v]$, $\quad V' = V \cap [u, v]$

Claim $c = \sup U'$ belongs to both U' and V' - Contradiction!

- c - limit point of U' $\Rightarrow c \in U'$

  U' - is closed in $[u, v]$

- $\forall\, x \in (c, v)$ in V' $\Rightarrow$ seq. in V' converging to c

  $\Rightarrow c$ = limit in V' $\Rightarrow c \in V'$. $\boxed{\checkmark}$

Intermediate value theorem

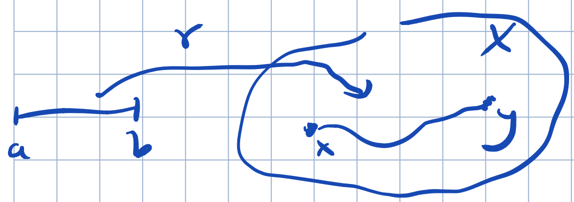
Let X be a connected space and $f: X \to \mathbb{R}$ a cont map

If $a, b \in f(X)$ $\forall c$ s.t. $a < c < b$ is in $f(X)$.

Proof assume $c \notin f(X)$

$$X = f^{-1}(\underset{\overset{\cup}{a}}{-\infty, c}) \sqcup f^{-1}(\underset{\overset{\cup}{b}}{c, +\infty}) \qquad \square$$

def: A top. space X is called path connected if $\forall x, y \in X$ $\exists$ path connecting x and y in X i.e. $\exists \gamma : [a,b] \xrightarrow{\text{cont}} X$ with $\gamma(a) = x$, $\gamma(b) = y$



Lemma Any path connected space X is connected.

Proof: assume X path connected but not connected, $X = U \sqcup V$

$\exists \gamma : [a,b] \to X$ $\quad \underset{U}{\overset{\cap}{\gamma(a)}}, \underset{V}{\overset{\cap}{\gamma(b)}} \Rightarrow [a,b] = \underset{\overset{\cup}{a}}{\gamma^{-1}U} \sqcup \underset{\overset{\cup}{b}}{\gamma^{-1}V}$

$\Rightarrow [a,b]$ is not connected – contradiction $\square$