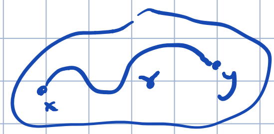
LAST TIME: **connected** space: $X \neq U_1 \amalg U_2$ - disj. union of non-empty open subsets

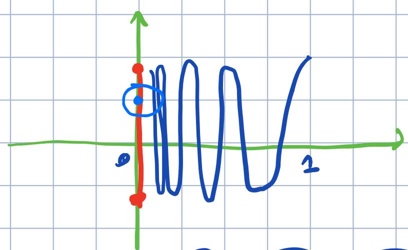
path connected: 

- intervals are connected
- path connected $\Longrightarrow$ connected

example of a space which is connected but not path connected:

topologist's sine curve

$$X = \{ (x, \sin\tfrac{1}{x}) \in \mathbb{R}^2 \mid 0 < x < 1 \} \cup \{ (0,y) \mid -1 \leq y \leq 1 \}$$

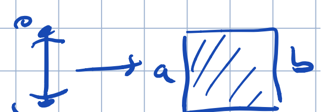


- **CW complexes**
  (cell complexes)
  C: closure-finite
  W: weak topology

def: A CW complex is a Hausdorff top space X and a collection of subsets $e_\alpha \subset X$ (cells) with following properties

(i) $X = \bigcup_\alpha e_\alpha$ $\qquad e_\alpha \cap e_\beta = \emptyset$ for $\alpha \neq \beta$

(ii) each cell is equipped with a characteristic map $\varphi_\alpha : D^{k} \to X$ s.t. (k: dimension of the cell, $k \geq 0$)

- $\varphi|_{\mathring{D}^k}$ is a homeomorphism from $\mathring{D}^k$ onto $e_\alpha \subset X$ 
- the boundary ∂D^k is sent to $(k-1)$-skeleton of X

$X_n = \bigcup_{\substack{\alpha \\ \text{s.t.} \\ \dim e_\alpha \leq n}} e_\alpha$ "n-skeleton" $\qquad \varphi_\alpha(\partial D^k) \subset X_{k-1}$

guarantee for finite CW complexes:

(iii) Each closure $\overline{e_\alpha} = \varphi_\alpha(D^k)$ is contained in the union of finitely many cells

(iv) a set $Y \subset X$ is closed in X iff $Y \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for all cells e_α.

$\varphi_\alpha|_{\partial D^k}$ is called the attaching map for e_α.

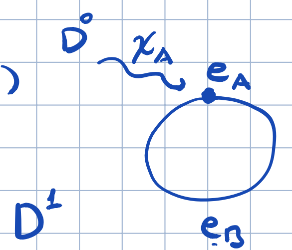
$$X_0 \subset X_1 \subset X_2 \subset \dots \subset X \qquad X = \bigcup_k X_k$$

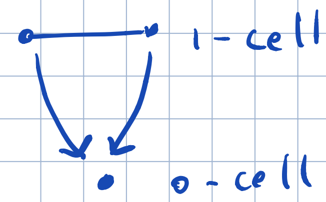
a subcollection of cells $Y = \bigcup_\beta e_\beta$ $\{\beta\} \subset \{\alpha\}$ is a "subcomplex" of X if $\overline{e_\beta} \subset Y$ $\forall e_\beta$.

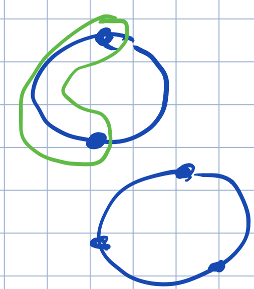
One builds a CW cx inductively: X_0 = collection of points (0-cells)

$$X_k = X_{k-1} \amalg \coprod_\alpha D^k_{(\alpha)} \Big/ x \in \partial D^k_{(\alpha)} \sim \varphi_\alpha(x) \in X_{k-1}.$$

Examples: (1)



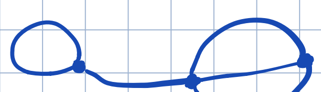




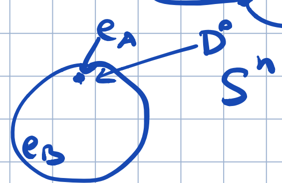
regular CW complex: $\varphi_\alpha : D^k \xrightarrow[\text{homeo}]{} \overline{e_\alpha}$

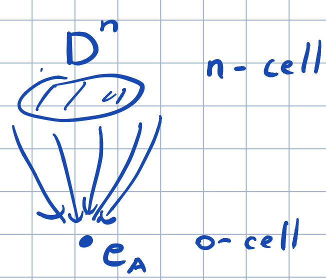
$\Leftrightarrow$ $\varphi_\alpha|_{\partial D^k} : \partial D^k \to \dots$ homeo onto its image

(2) a finite graph = 1-dim CW complex



(3)

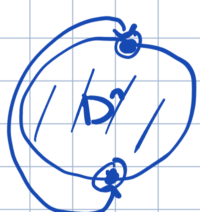




(4) $\mathbb{RP}^n = D^n \cup_{\varphi : S^n \xrightarrow[2\to 1]{} \mathbb{RP}^{n-1}} \mathbb{RP}^{n-1}$

by induction, $\mathbb{RP}^n$ is a CW cx with a single cell in each dim $0, 1, \dots, n$

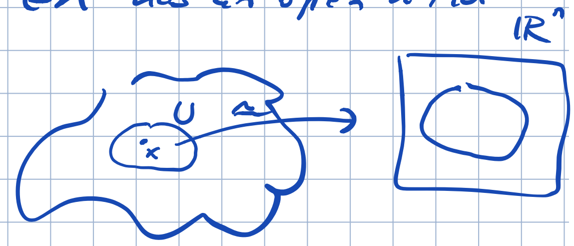
$\mathbb{RP}^n = e^0 \cup e^1 \cup \dots \cup e^n$.



$$D^n \downarrow \mathbb{RP}^{n-1}$$

Topological manifolds

def A manifold of dimension n (n-mfd) is a top space which is locally homeo to $\mathbb{R}^n$, i.e. $\forall x \in X$ has an open nbhd $\underset{\substack{\Psi \\ x}}{U} \subset X$ s.t. $U \underset{\text{homeo}}{\approx} \underset{\text{open}}{V \subset} \mathbb{R}^n$



Additionally, X is assumed to be
- Hausdorff
- second countable ($\exists$ countable basis of topology).

Rem: If X is Hausdorff then any subspace $Y \subset X$ is also H.

If X is 2^{nd} countable ———— // ———— 2^{nd} countable

Ex: Any open $U \subset \mathbb{R}^n$ is a manifold

- $S^n \subset \mathbb{R}^{n+1}$ is an n-manifold