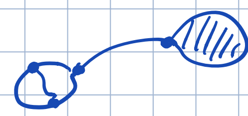
LAST TIME:

- CW complex: 
  (cell)

- (topological) n-manifold: X s.t. $\forall x \in X$ has an open nbhd homeo to an open in $\mathbb{R}^n$

  (+ Hausdorff, + 2nd countable)

- Ex: any open $U \subset \mathbb{R}^n$ is an n-manifold.

$$S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$$

$$U_i^+ = \{ x = (x_0, \dots, x_n) \mid x_i > 0 \} \quad , \quad i = 0 \dots n$$

$$U_i^- = \{ \text{———} \quad '' \quad \text{———} \; x_i < 0 \}$$

$$U_i^\pm \xrightarrow{1-1} \mathring{D}^n = \{ (v_1, \dots, v_n) \mid v_1^2 + \dots + v_n^2 < 1 \} \quad \text{homeomorphism}$$

$$x \longmapsto (x_0 \dots \hat{x}_i \dots x_n)$$

$$(v_1 \dots \underset{\underset{i}{\uparrow}}{\pm\sqrt{1-\|v\|^2}} \dots v_n) \longleftarrow (v_1 \dots v_n)$$

- If $\underset{\underset{m\text{-dim}}{\uparrow}}{X}, \underset{\underset{n\text{-dim}}{\uparrow}}{Y}$ -mfds $\qquad X \times Y$ $(m+n)$-dim mfd

- $\mathbb{RP}^n$ is an <u>n-mfd.</u>

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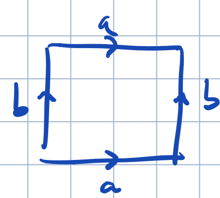
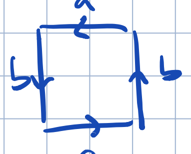
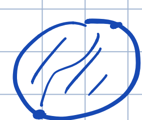
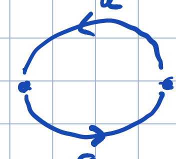
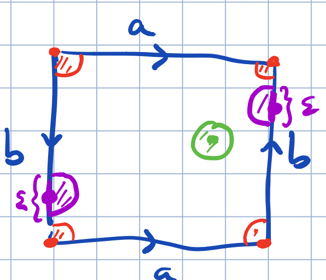
dim = 2 manifolds

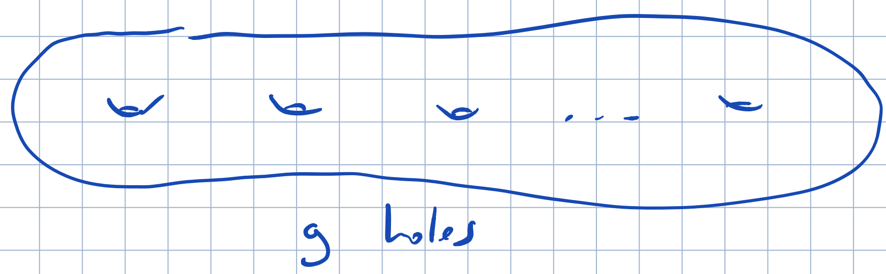
- 2-torus $\quad T \approx$  $\approx S^1 \times S^1$ - 2-manifold

- $\mathbb{RP}^2 \approx$  $\approx$  $\approx$ 

- Klein bottle $\approx$  - manifold
  K

$$K \approx \mathbb{RP}^2 \# \mathbb{RP}^2$$
connected sum

• genus g surface Σ_g - subspace of $\mathbb{R}^3$



Σ_1 = torus

Σ_0 = sphere

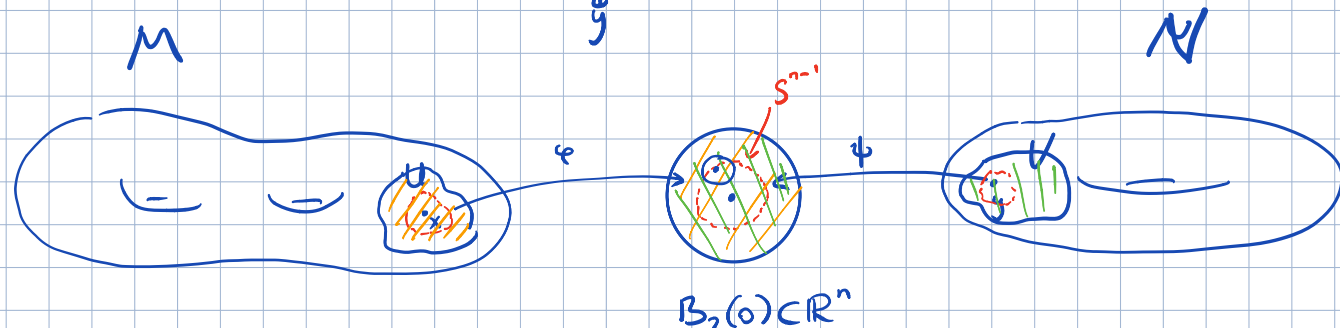
Connected sum construction

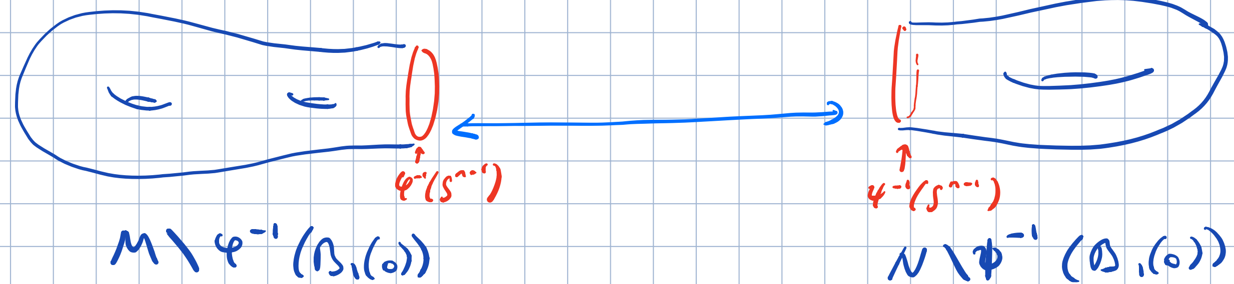
M, N - n - mfds $\rightsquigarrow$ $M \# N$ - n - mfd (new) "connected sum"

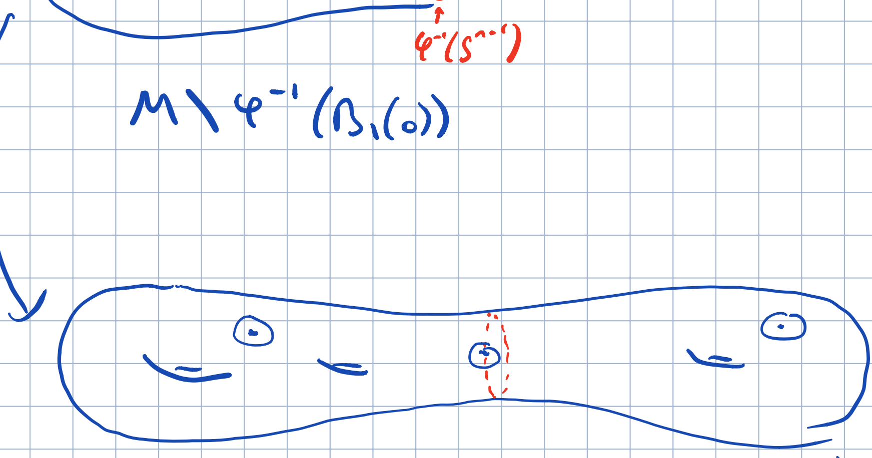
• choose $x \in M$, $y \in N$

pick a homeo $\varphi : \underset{x}{\overset{\cup}{U}} \subset M \xrightarrow{\sim} B_2(0) \subset \mathbb{R}^n$ (open)

$\psi : \underset{y}{\overset{\cup}{V}} \subset N \xrightarrow{\sim} B_2(0)$







$$M \# N := M \setminus \varphi^{-1}(B_1(0)) \cup N \setminus \psi^{-1}(B_1(0)) \Big/ \varphi^{-1}(S^{n-1}) \sim \psi^{-1}(S^{n-1})$$

$$z \longmapsto \psi^{-1}(\varphi(z))$$

- $M \# N$ - is a manifold
- dependence on choice of x, y; φ, ψ:
  should assume M, N connected

<u>Fact</u>: the result is unique up to homeo if one is "care with orientations"

in $\dim = 2$, the result is unique.

$$\text{Ex:} \quad \Sigma_2 \# T \approx \Sigma_3 \quad , \quad \Sigma_g \# \Sigma_{g'} \approx \Sigma_{g+g'}$$

$$\Rightarrow \underbrace{T \# \cdots \# T}_{g} \approx \Sigma_g \quad (\text{take it to be the def of } \Sigma_g)$$

$$X_k := \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{k} \quad - \text{"}k\text{-fold projective plane"}$$

<u>Classification Theorem for compact, connected 2-mfds</u>

Every cpt, conn 2-mfd is homeo to <u>exactly</u> one of the following.

- genus g surface $\Sigma_g = \underbrace{T \# \cdots \# T}_{g \geq 1}$ or $\Sigma_0 = S^2$
  $g \geq 0$
- $X_k = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{k}$, $k \geq 1$

Euler char. + orientability

(1) 2-mfds $\Sigma_0, \Sigma_1, \Sigma_2, \ldots$, $X_1, X_2, \ldots$ are pairwise non-homeo

(2) any cpt conn 2-mfd is homeo to one of the std ones

Rem: Klein bottle $\approx \mathbb{RP}^2 \# \mathbb{RP}^2 = X_2$

$\mathbb{RP}^2 \# T \approx \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 = X_3$

<u>Euler characteristic</u>

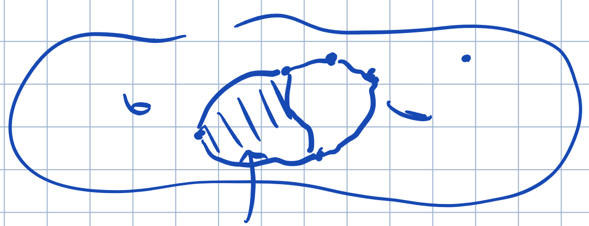
Let Σ be a cpt 2-mfd. A cell (CW) decomp γ of Σ "a pattern of polygons"

- graph Γ on Σ — a coll. of point (vertices) $v_1 \dots v_k \in \Sigma$
0-cells

paths (edges) $e_1 \dots e_l : [0,1] \to \Sigma$
1-cells

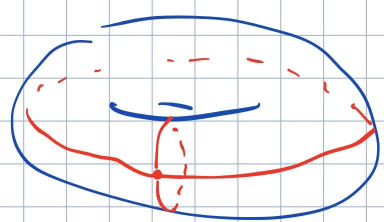
endpoints belong to $V = \{v_1 \dots v_k\}$

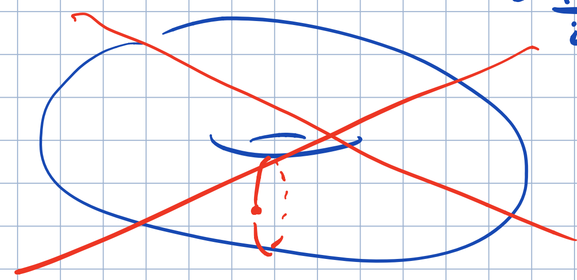
intersections - only at endpoints



a polygon

such that $\Sigma \setminus \Gamma \approx \coprod_i \mathring{D}^2$

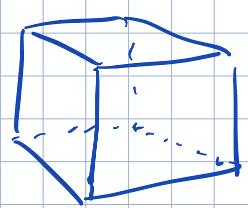
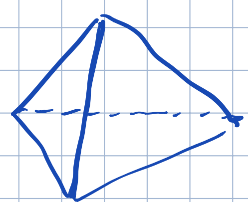




Euler char $\quad \chi(\Sigma; \gamma) = \#\text{ vertices} - \#\text{ edges} + \#\text{ polygons}$

$$\chi = \sum_{n \geq 0} (-1)^n \#\, n\text{-cells}$$

for a finite CW cx

$S^2 \simeq$  $\simeq$ 

$8 - 12 + 6 = 2 \qquad 4 - 6 + 4 = 2$