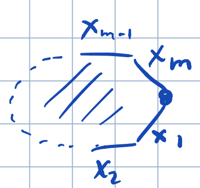
Proposition Let M, N be two cpt, conn 2-mfds, $M = \Sigma(W_1)$, $N = \Sigma(W_2)$, W_1, W_2 - words from disjoint alphabets. Then the connected sum is

$$M \# N = \Sigma(W_1 W_2)$$

Proof $W_1 = x_1 \cdots x_m$ $\qquad W_2 = y_1 \cdots y_n$

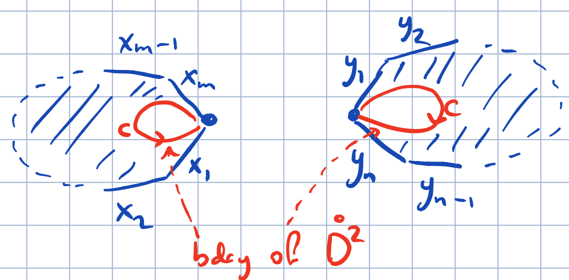
$M = \Sigma(W_1) =$



$= \Sigma(W_2) = N$

remove $\mathring{D}^2$ from M and N

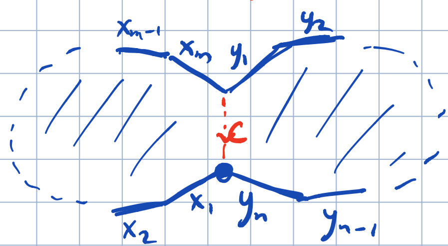
$M \setminus \mathring{D}^2 =$



$= N \setminus \mathring{D}^2$

glue along the circle c

$M \# N =$



$$\Rightarrow M \# N = \Sigma(x_1 \cdots x_m y_1 \cdots y_n) = \Sigma(W_1 W_2) \qquad \square$$

Corollary: (1) $\Sigma_g = \underbrace{T\# \cdots \# T}_{g} \approx \Sigma(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1})$

$\approx \Sigma(a_1 b_1 a_1^{-1} b_1^{-1}) \# \cdots \# \Sigma(a_g b_g a_g^{-1} b_g^{-1})$

(2) $X_k = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{k} \approx \Sigma(a_1 a_1 a_2 a_2 \cdots a_k a_k)$

$\approx \Sigma(a_1 a_1) \# \cdots \# \Sigma(a_k a_k)$

Proposition B: Let W_1, W_2, W_3 be words and a a letter not occurring in them. Then there are homeomorphisms

(*) $$\Sigma(W_1 a W_2 a W_3) \approx \Sigma(W_1 a a W_2^{-1} W_3),$$

(**) $$\Sigma(W_1 a W_2 a W_3) \approx \Sigma(W_1 W_2^{-1} a a W_3)$$

where W_2^{-1} is the "inverse" of the word W_2: $W_2 = x_1 \cdots x_n \to W_2^{-1} = x_n^{-1} \cdots x_1^{-1}$.
(as for a group product)

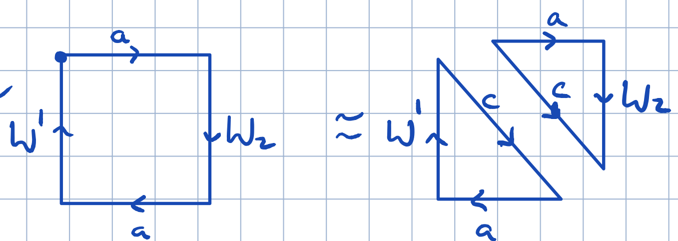
Proof: <let's check (*)>

$$\Sigma(W_1 a W_2 a W_3) \underset{(@)}{\approx} \Sigma(a W_2 a \underbrace{W'}_{W_3 W_1})$$

$$\Sigma(W_1 a a W_2^{-1} W_3) \approx \Sigma(a a W_2^{-1} W')$$

So, we want to prove: $\Sigma(a W_2 a W') \approx \Sigma(a a W_2^{-1} W')$

$\Sigma(a W_2 a W') \approx$



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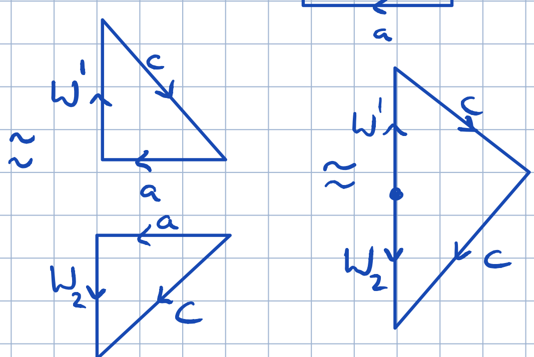

$\approx \Sigma(c c W_2^{-1} W') \approx \Sigma(a a W_2^{-1} W')$

(**) is similar - we cut the square by the other diagonal $\quad \square$

Application: proof of $T \# \mathbb{RP}^2 \approx \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

Indeed: $T \# \mathbb{RP}^2 = \Sigma(aba^{-1}b^{-1}) \# \Sigma(cc) \approx \Sigma(aba^{-1}b^{-1}cc)$

$\approx \Sigma(abcbac) \approx \Sigma(abbc^{-1}ac) \approx \Sigma(bbc^{-1}aca)$

$\approx \Sigma(bbc^{-1}c^{-1}aa) \approx \Sigma(bb) \# \Sigma(c^{-1}c^{-1}) \# \Sigma(aa) \approx \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

$\boxtimes$

- Idea of proof of "constructive part" of the classification THM
  <that any cpt conn 2-mfd is $\approx \Sigma_g$ or X_k>

(1) show that every Σ admits a triangulation (cell decomp where polygons = triangles)
→ give each edge its own label

$\Rightarrow \Sigma = \amalg$ polygons / gluing edges carrying same label

(2) Reduce the number of polygons by one by gluing a pair of edges with same label belonging to different polygons

⟶ inductively, reduce to a single polygon

(3) Use the moves of Lemma A, Prop. B to show that labeling of edges of the polygon can be modified, without changing the homeo type of the resulting quotient, to obtain the stand. labeling for Σ_g or X_k.

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def A 2-mfd Σ is non-orientable if it contains a subspace homeo to the Möbius band. Otherwise, Σ is called orientable.

Prop (i) X_k is non-orientable

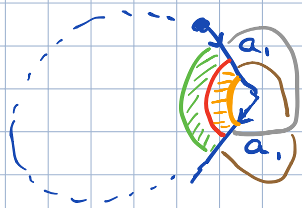
(ii) Σ_g is orientable.

Rem: if $\Sigma \overset{f}{\approx} \Sigma'$ a homeo of 2-mfds then both are either orientable or non-orientable.

(if $M \subset \Sigma$ homeo to M.b., then $f(M) \subset \Sigma'$ is too)

Thus, Prop $\Rightarrow \Sigma_g \not\approx X_k$ for any g, k !

Proof of (i): $X_k = \Sigma(a_1 a_1 \cdots a_k a_k)$



after gluing, bi-colored strip becomes Möbius strip.
green part gets glued to orange part.

$\Rightarrow$ X_k contains a Möbius strip!

<sketch>

(ii) Let $i : M \longrightarrow \Sigma_g$ homeo onto its image

$M = [0,1]\times(-1,1) / (0,t)\sim(1,-t)$

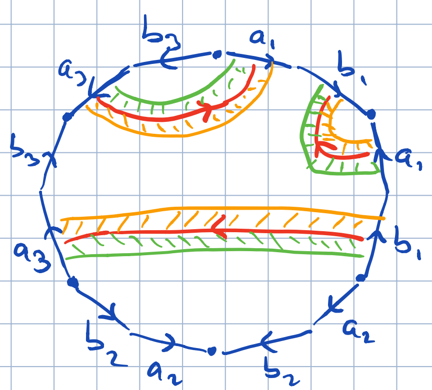
open mobius strip

$\leftarrow$ $C = [0,1]\times\{0\}/\sim$ central circle

recognition principle for the Möbius band

Lemma: Any open nbhd $U \subset M$, $U \supset C$, contains a sub-nbhd $V \subset U$, $V \supset C$, s.t. $V \setminus C$ is path-connected.

$\Sigma_g = \Sigma(a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1})$



$4g$-gon

red curve $= i(C)$

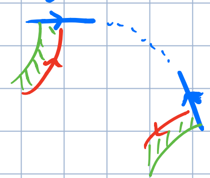
its nbhd = 2-sided strip U

$U \setminus i(C)$ is green part $\amalg$ red part - disjoint!

- Contradiction with Lemma

green part to the right as we go along $i(C)$.

This is consistent with gluing because ℓ is always glued to ℓ^{-1}!



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