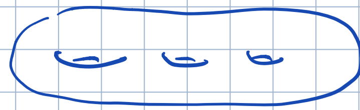
Quiz 2, 8/31/20

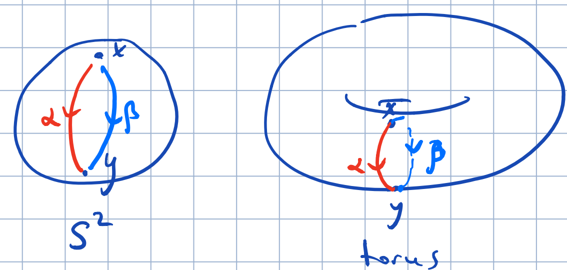
- Consider the surface $\Sigma =$  $\#$ Klein bottle

(a) what is the Euler characteristic?

(b) is it orientable?

(c) to which of the "standard" surfaces Σ_g, X_k is it homeomorphic? [Hint: use the answers for (a), (b)]

Fundamental group

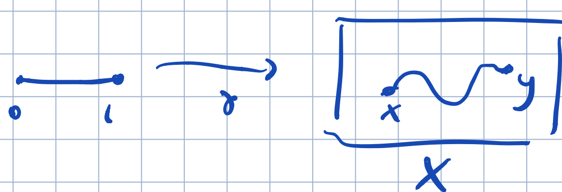


X paths on $X / \sim$

def: A path in X is a cont. map $\gamma : [0,1] \to X$

$\gamma(0) = x$

$\gamma(1) = y$

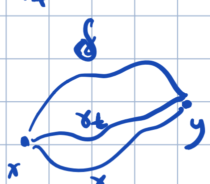


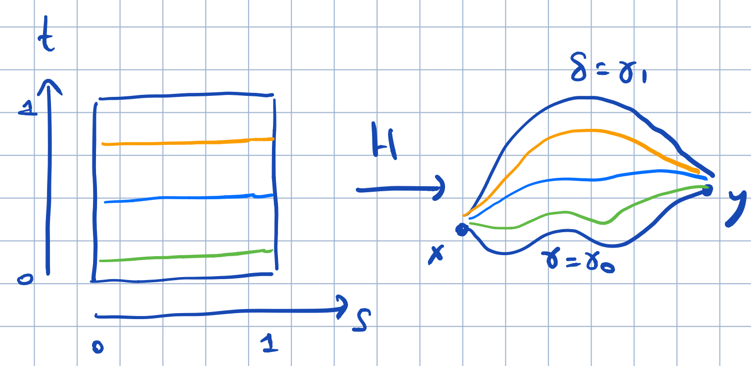
- Let γ, δ two paths in X from x to y.

These paths are homotopic relative to endpoints

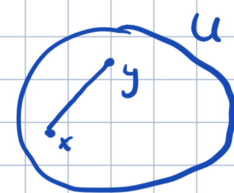
if $\forall t \in [0,1]$ $\exists \gamma_t$ a path from x to y s.t.

- $\gamma_0 = \gamma$, $\gamma_1 = \delta$
- the map $H : [0,1] \times [0,1] \to X$ is continuous
$(s, t) \longmapsto \gamma_t(s)$



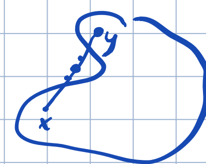


Let $U \subset \mathbb{R}^n$ be a convex set



$\forall x, y$ in U, the straight line segment

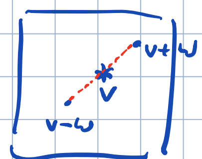
$\{ \underbrace{(1-t)x + ty}_{\in \mathbb{R}^n} \mid t \in [0,1] \} \subset U$



Ex: • $\mathbb{R}^n$

- an open ball $B_r(x)$
- a closed ball $D_r(x)$
- a non-example $\mathbb{R}^n \setminus \{v\}$

not convex!



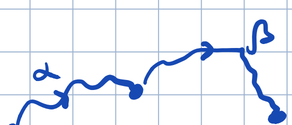
$\alpha \sim \beta$ equivalence relation!

Lemma (*): Let $U \subset \mathbb{R}^n$ convex subset, let α, β be two paths from x to y.

Then $\alpha \sim \beta$, with explicit (linear) homotopy

$$H : [0,1] \times [0,1] \longrightarrow U$$
$$(s, t) \longmapsto (1-t)\alpha(s) + t\beta(s)$$

def Let $\alpha, \beta : I \to X$ two paths, $\boxed{\alpha(1) = \beta(0)}$,

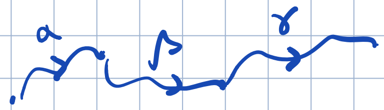


Then we can form a new path $\alpha * \beta$ - the concatenation of α and β - by first following α and then β.

$$\alpha * \beta : I \to X$$
$$s \mapsto \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

Associativity:

$\alpha, \beta, \gamma \to$ paths $\qquad \alpha(1) = \beta(0), \quad \beta(1) = \gamma(0)$



$$\alpha * (\beta * \gamma) \neq (\alpha * \beta) * \gamma$$

$$\begin{array}{ccc} \alpha & \beta & \gamma \\ \downarrow & \downarrow & \downarrow \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{array} \qquad \begin{array}{ccc} \alpha & \beta & \gamma \\ \downarrow & \downarrow & \downarrow \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array}$$

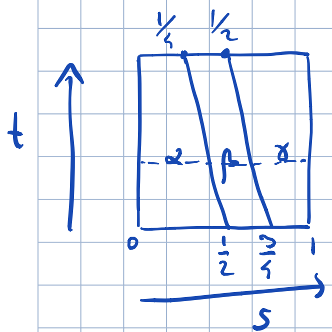
<u>Lemma</u> (**): concatenation is assoc. up to homotopy:

$$\underbrace{\alpha * (\beta * \gamma)}_{\delta} \sim \underbrace{(\alpha * \beta) * \gamma}_{\varepsilon}$$

<u>Hatcher</u>

$\varepsilon = \delta \circ \varphi$

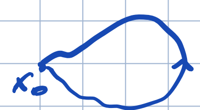
[ ] - class up to homotopy $\quad [\alpha * (\beta * \gamma)] = [(\alpha * \beta) * \gamma]$



$$H(s,t) = \begin{cases} \alpha\left(\dfrac{s}{\frac{1}{2} - \frac{1}{4}t}\right), & 0 \le s \le \frac{1}{2} - \frac{1}{4}t \\ \beta\left(4\left(s - \frac{1}{2} + \frac{1}{4}t\right)\right), & \frac{1}{2} - \frac{1}{4}t \le s \le \frac{3}{4} - \frac{1}{4}t \\ \gamma\left(1 - \dfrac{1-s}{\frac{1}{4} + \frac{1}{4}t}\right), & \frac{3}{4} - \frac{1}{4}t \le s \le 1 \end{cases}$$

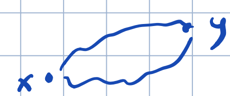
<u>way 1</u>

pick $x_0 \in X$



fundamental - group

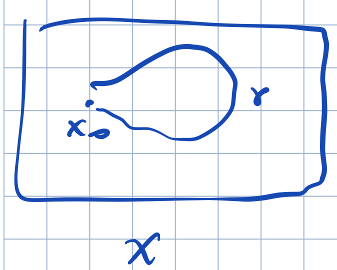
<u>way 2</u>



fundamental groupoid

* <u>def</u> Let X a top space, $x_0 \in X$ - "base point"

(X, x_0) - "pointed top. space". A <u>based loop</u> in (X, x_0) is a path $\gamma : I \to X$ with $\gamma(0) = x_0 = \gamma(1)$



Let $\pi_1(X, x_0) = \{\text{based loops in } (X, x_0)\} / \text{homotopy.}$

<u>Proposition</u> The set $\pi_1(X, x_0)$ is a <u>group</u> - the fundamental group of (X, x_0)

with

- multiplication $[\alpha] \cdot [\beta] = [\alpha * \beta]$ - associative (from Lemma **)

• identity $[c_{x_0}]$

$\uparrow$

$c_{x_0}: I \to X$

$s \mapsto x_0$

• inverse of an element $[\gamma]$ is $[\bar{\gamma}]$

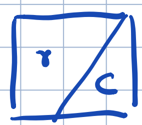
$\bar{\gamma}: I \to X$

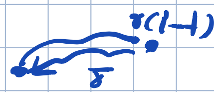
$s \mapsto \gamma(1-s)$

$\Leftarrow$ lemma

lemma: $\gamma: I \to X$, $\bar{\gamma}$ - inverse, c_x constant path at x

$\gamma * c_{\gamma(1)} \sim \gamma$, $c_{\gamma(0)} * \gamma \sim \gamma$, $\gamma * \bar{\gamma} \sim c_{\gamma(0)}$, $\bar{\gamma} * \gamma \sim c_{\gamma(1)}$





$$H(s,t) = \begin{cases} \gamma(2s), & 0 \le s \le \frac{1-t}{2} \\ \gamma(1-t), & \frac{1-t}{2} \le s \le \frac{1+t}{2} \\ \gamma(2-2s), & \frac{1+t}{2} \le s \le 1 \end{cases}$$

$\tilde{\gamma}(2s-1)$