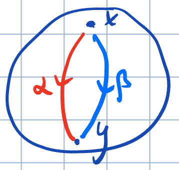
# Fundamental group

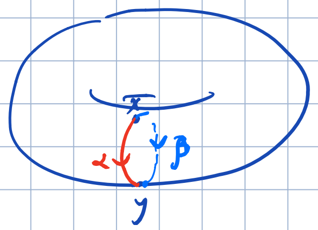
Idea: study closed paths $\gamma$ on $\underset{\text{top space}}{X}$ / $\gamma_0 \sim \gamma_1$ if "$\gamma_0$ can be deformed into $\gamma_1$"

- paths on different top spaces have different behavior:



$S^2$

$\alpha$ can be deformed into $\beta$

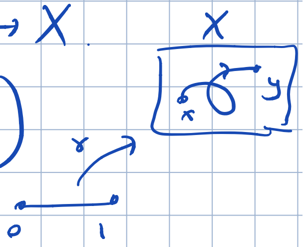


torus

$\alpha$ cannot be deformed into $\beta$.

* <u>def</u> A <u>path</u> in a top space $X$ is a cont. map $\gamma : [0,1] \to X$.

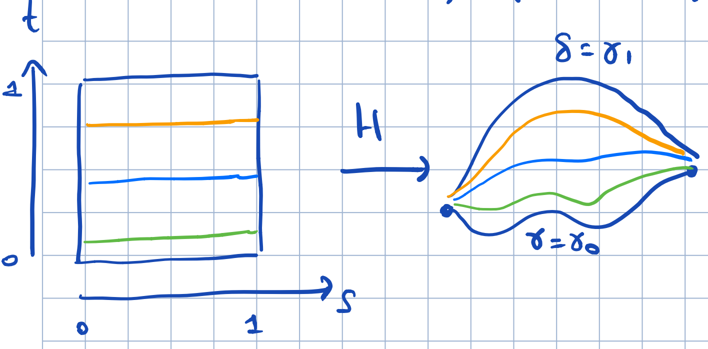
( $\gamma(0)$ - starting point, $\gamma(1)$ - endpoint; $\{\gamma(0), \gamma(1)\}$ - endpoints
$\gamma(0) = x$, $\gamma(1) = y$ $\rightsquigarrow$ path is from $x$ to $y$ )



* Let $\gamma, \delta$ be two paths in $X$ from $x$ to $y$.
These paths are "homotopic relative to endpoints" (or "path homotopic", or just "homotopic")
if $\forall t \in [0,1]$ $\exists$ path $\gamma_t$ from $x$ to $y$ s.t.

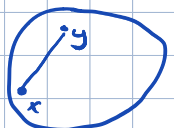
- $\gamma_0 = \gamma$, $\gamma_1 = \delta$
- the map $H : [0,1] \times [0,1] \to X$ is <u>continuous</u>
  $(s, t) \mapsto \gamma_t(s)$

(i.e. the family of paths $\gamma_t$ depends continuously on the parameter $t$ )



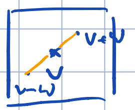
H - "homotopy" between $\gamma$ and $\delta$
we write $\underbrace{\gamma \sim \delta}$ "homotopic"
$\sim$ is an equivalence relation (check it!)
$[\gamma]$ - homotopy class of a path $\gamma$.

* Let $U \subset \mathbb{R}^n$ be a <u>convex</u> subset, i.e. $\forall$ $x, y \in U$, the straight line segment between $x$ and $y$ $= \{(1-t)x + ty \in \mathbb{R}^n \mid t \in [0,1]\}$ is in $U$.



Ex:
- $\mathbb{R}^n$
- an open ball $B_r(x)$
- a closed ball $D_r(x)$

∴ <u>non-example</u>: $\mathbb{R}^n \setminus \{v\}$ is not convex! $\forall \underset{w \neq 0}{w \in \mathbb{R}^n}$, line segment between $v+w$ and $v-w$ contains $v$.

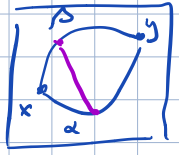


②

**Lemma** (*) Let $U \subset \mathbb{R}^n$ convex, let $\alpha, \beta$ be paths in $U$ with same endpoints $\alpha(0) = \beta(0) = x$, $\alpha(1) = \beta(1) = y$.

Then $\alpha \sim \beta$ with an explicit ("linear") homotopy

$$H : [0,1] \times [0,1] \longrightarrow U$$
$$(s\ ,\ t) \longmapsto (1-t)\,\alpha(s) + t\,\beta(s)$$

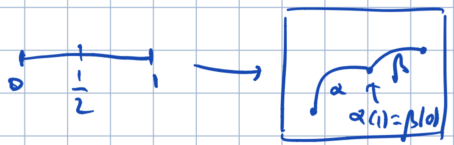


- note: for $s \in [0,1]$, the path $t \mapsto H(s,t)$ is a straight line path from $\alpha(s)$ to $\beta(s)$

* **def** Let $\alpha, \beta : \underbrace{I}_{[0,1]} \to X$ be paths in $X$ s.t. $\alpha(1) = \beta(0)$. Then we can form a new path $\alpha * \beta$ - the concatenation of $\alpha$ and $\beta$ - by first following $\alpha$ and then $\beta$.

Explicitly:

$\alpha * \beta : I \to X$ is given by $$\boxed{\alpha * \beta(s) = \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}}$$



* Let $\alpha, \beta, \gamma$ be paths in $X$ with $\alpha(1) = \beta(0)$, $\beta(1) = \gamma(0)$

$\alpha * (\beta * \gamma)\ (s)$ is on $\alpha$ for $0 \le s \le \frac{1}{2}$, on $\beta$ for $\frac{1}{2} \le s \le \frac{3}{4}$, on $\gamma$ for $\frac{3}{4} \le s \le 1$.

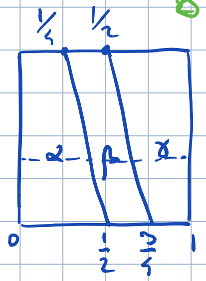
$(\alpha * \beta) * \gamma\ (s)$ —— " —— $0 \le s \le \frac{1}{4}$ —— " —— $\frac{1}{4} \le s \le \frac{1}{2}$ —— " —— $\frac{1}{2} \le s \le 1$

So, generally $\alpha * (\beta * \gamma) \neq (\alpha * \beta) * \gamma$

**Lemma** ** Concatenation is associative up to homotopy: if $\alpha, \beta, \gamma$ paths in $X$ with $\alpha(1) = \beta(0)$, $\beta(1) = \gamma(0)$, then

$$\underbrace{\alpha * (\beta * \gamma)}_{\delta} \sim \underbrace{(\alpha * \beta) * \gamma}_{\varepsilon} \qquad \text{or equivalently} \quad [\alpha * (\beta * \gamma)] = [(\alpha * \beta) * \gamma]$$

Proof:



$$H(s,t) = \begin{cases} \alpha\left(\dfrac{s}{\frac{1}{2} - \frac{1}{4}t}\right), & 0 \le s \le \frac{1}{2} - \frac{1}{4}t \\ \beta\left(4\left(s - \frac{1}{2} + \frac{1}{4}t\right)\right), & \frac{1}{2} - \frac{1}{4}t \le s \le \frac{3}{4} - \frac{1}{4}t \\ \gamma\left(1 - \dfrac{1-s}{\frac{1}{4} + \frac{1}{4}t}\right), & \frac{3}{4} - \frac{1}{4}t \le s \le 1 \end{cases}$$

Hatcher:

$\gamma_1 = \gamma_0 \circ \varphi$ ← reparametrization

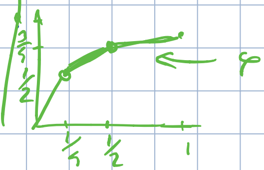
$\varphi : I \to I$, $\varphi(0) = 0$, $\varphi(1) = 1$

then $\gamma_1 \sim \gamma_0$ via

$\gamma_t = \gamma_0 \circ ((1-t)s + t\varphi(s))$ $\boxtimes$

In our case,

$\varepsilon = \delta \circ \varphi$ with



- want to construct a group out of homotopy classes of paths

problem: need matching endpoints for concatenation.

- **Way 1** pick $x_0 \in X$ and consider only paths that start and end at $x_0$
  $\langle \rightsquigarrow$ fund. group of $X \rangle$
- **Way 2** give up the idea of a group and construct instead a groupoid - the fundamental groupoid of $X$.

* def: Let $X$ be a top space and $x_0 \in X$ (a "base point").
Such a pair $(X, x_0)$ is called a "pointed topological space". A based loop in $(X, x_0)$ is a path $\gamma : I \to X$ with $\gamma(0) = x_0 = \gamma(1)$.

Let $\pi_1(X, x_0) = \{\text{based loops in } (X, x_0)\} / \text{homotopy}$

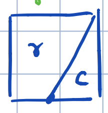
Proposition: The set $\pi_1(X, x_0)$ is a group, the fundamental group of $(X, x_0)$ with

- multiplication given by concatenation of based loops
  $[\alpha] \cdot [\beta] = [\alpha * \beta]$ for based loops $\alpha, \beta$. ← assoc. follows from Lemma**
- identity element of $\pi_1(X, x_0)$ is the homotopy class of the constant path $c_{x_0}$ ($c_{x_0}(s) = x_0$, $0 \le s \le 1$).
- inverse of an element $[\gamma] \in \pi_1(X, x_0)$ is given by $[\tilde\gamma]$ where $\tilde\gamma : I \to X$ is the path $\gamma$ run backwards, i.e. $\tilde\gamma(s) = \gamma(1-s)$.

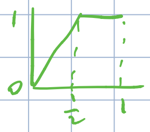
} from Lemma below

Lemma Let $\gamma : I \to X$ a path, let $\tilde\gamma : I \to X$ be the path $\tilde\gamma(s) = \gamma(1-s)$ and let $c_x : I \to X$ be the constant path at $x \in X$. Then there are homotopies

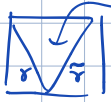
$$\underbrace{\gamma * c_{\gamma(1)}}_{\delta} \sim \gamma, \quad c_{\gamma(0)} * \gamma \sim \gamma, \quad \gamma * \tilde\gamma \sim c_{\gamma(0)}, \quad \tilde\gamma * \gamma \sim c_{\gamma(1)}.$$



or via reparam: $\delta = \gamma \circ \varphi$



$$H(s,t) = \begin{cases} \gamma(2s), & 0 \le s \le \frac{1-t}{2} \\ \gamma(1-t), & \frac{1-t}{2} \le s \le \frac{1+t}{2} \\ \gamma(2-2s), & \frac{1+t}{2} \le s \le 1 \\ \tilde\gamma \text{''}(2s-1) & \end{cases}$$



stationary paths

stopped here

Ex: Let $X$ be a convex subset of $\mathbb{R}^n$ and $x_0 \in X$. Then (Lemma *) any based loop in $X$ is homotopic to $c_{x_0}$ $\Rightarrow$ group $\pi_1(X, x_0)$ is trivial.

Lemma Let $X$ be a top. space and $\beta$ a path from $x_0$ to $x_1$. Then the map

$$\Phi : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1) \qquad [\gamma] \longmapsto [\tilde\beta * \gamma * \beta]$$

is an isomorphism of groups.

In particular, the isomorphism class of $\pi_1(X, x_0)$ of a path connected space does not depend on the choice of base point $x_0 \in X$.

Proof: • for $\{\gamma_t\}$ a homotopy of loops based at $x_0$, $\{\tilde\beta * \gamma_t * \beta\}$ - a homotopy of loops based at $x_1$
$\Rightarrow$ $\Phi$ well-defined

- $\Phi([\gamma_1])\Phi([\gamma_2]) = [\tilde\beta * \gamma_1 * \beta]\cdot[\tilde\beta * \gamma_2 * \beta] = [\tilde\beta * \gamma_1 * \underbrace{\beta * \tilde\beta}_{\sim c_{x_0}} * \gamma_2 * \beta]$

  $= [\tilde\beta * (\gamma_1 * \gamma_2) * \beta] = \Phi([\gamma_1]\cdot[\gamma_2]) \Rightarrow \Phi$ homomorphism

- $\Phi$ is an iso with inverse $\Phi' : \underset{\pi_1(X,x_1)}{\underset{\cap}{[\gamma']}} \longmapsto [\beta * \gamma' * \tilde\beta]$ :

  $\Phi' \circ \Phi : [\gamma] \longmapsto \Phi'([\tilde\beta * \gamma * \beta]) = [\underbrace{\beta * \tilde\beta}_{\sim c_{x_0}} * \gamma * \underbrace{\beta * \tilde\beta}_{\sim c_{x_0}}] = [\gamma]$ ☑

- A space $X$ is "simply connected" if it is path connected <u>and</u> $\pi_1(X, \underset{\substack{\uparrow \\ \text{any point}}}{x_0}) = 0$

**Lemma** $X$ is simply connected iff $\forall x,y \in X$ there exists a unique homotopy class of paths $x \to y$.

**Proof** existence of a path $\Leftrightarrow X$ path connected

suppose $\pi_1(X) = 0$ and $\alpha, \beta$ two paths between $x, y$. Then $\alpha \sim \overbrace{\alpha * \tilde\beta}^{\sim c_y \text{ using } \pi_1 = 0} \!\! \underbrace{\phantom{\tilde\beta} * \beta}_{\sim c_x} \sim \beta$

Conversely: if there is a unique homotopy class of paths $x \to x$, then $\pi_1(X, x) = 0$. ☑