CORRECTION

- Let (X_1, x_1), (X_2, x_2) pointed top spaces, s.t. x_1, x_2 have contractible open nbhds
(or at least simply connected)

Let $X_1 \xrightarrow[j_1]{} X_1 \vee X_2 \xleftarrow[j_2]{} X_2$ be the inclusion maps

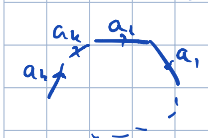
Then the map $\pi_1(X_1) * \pi_1(X_2) \longrightarrow \pi_1(X_1 \vee X_2)$

given by $c_1 \in \pi_1(X_1) \longmapsto (j_1)_*(c_1) \in \pi_1(X_1 \vee X_2)$

$c_2 \in \pi_1(X_2) \longmapsto (j_2)_*(c_2) \in \pi_1(X_1 \vee X_2)$

is an iso of groups.

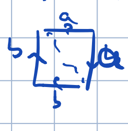
- $\pi_1(X_k) = \{a_1, \dots, a_k \mid a_1^2 \cdots a_k^2 = 1\}$

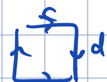


In particular: $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$

$$\pi_1(X_2) = \pi_1(K)$$

$$\{a, b \mid a^2 b^2 = 1\} \qquad \{c, d \mid c d c^{-1} d = 1\}$$





Products, coproducts, pushouts

Products

recall: for X_1, X_2 top spaces, $\begin{tikzcd} & X_1 \times X_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & & X_2 \end{tikzcd}$ projections.

$f : Y \to X_1 \times X_2$ is cont. iff
component maps $f_1 = p_1 \circ f : Y \to X_1$ and $f_2 = p_2 \circ f : Y \to X_2$ are cont.

Or: $f : Y \to X_1 \times X_2$ is uniquely determined by a pair $f_1 : Y \to X_1$, $f_2 : Y \to X_2$
and $f_i = p_i \circ f$

Diagrammatically:

$$\begin{tikzcd} & X_1 \\ Y \arrow[ur, bend left, "f_1"] \arrow[r, dashed, "\exists! f"] \arrow[dr, bend right, "f_2"'] & X_1 \times X_2 \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & X_2 \end{tikzcd}$$

- given the commut. diagram given by solid arrows, $\exists!$ map f - dashed arrow - making the whole diag. commutative.

(commutativity in top/bottom triangles $\Leftrightarrow$ f_1, f_2 are components of f)

In any category:

def Let X_1, X_2 be objects in a category C. $X \in Ob(C)$ is the "categorical product" (denoted $X_1 \times X_2$) if there are morphisms $p_1 : X \to X_1$, $p_2 : X \to X_2$ s.t. the diagram $X_1 \xleftarrow{p_1} X \xrightarrow{p_2} X_2$ has the property: $\forall\, Y \in Ob(C)$ and $f_i : Y \to X_i$, $i = 1,2$, $\exists!\, f : Y \to X$ making the diagram

$$\begin{tikzcd} & X_1 \\ Y \arrow[ur, "f_1"] \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_2"'] & X \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & X_2 \end{tikzcd}$$

$(*)$ commutative

Rem 1. Fact, "the" categorical product is defined up to (a unique) isomorphism:

$$\begin{tikzcd} & X_1 & \\ X \arrow[ur, "p_1"] \arrow[dr, "p_2"'] \arrow[rr, shift right, dashed, "g"'] & & X' \arrow[ul, "p_1'"'] \arrow[dl, "p_2'"] \arrow[ll, shift right, dashed, "f"'] \\ & X_2 & \end{tikzcd}$$

$f \circ g = id_X$ by uniqueness in $(*)$ for $Y = X$.

- for $C = Set, Vect, Grp, Top, Top_*$, the categorical product = Cartesian product with usual projection maps $p_i : X_1 \times X_2 \to X_i$ $\quad i = 1,2$

Coproducts.

motivating example: disjoint union of sets

- for X_1, X_2 sets, the disjoint union is the set

$$X_1 \amalg X_2 := \{(x,1) \mid x \in X_1\} \cup \{(x,2) \mid x \in X_2\} \subset (X_1 \cup X_2) \times \{1,2\}$$

$$X_1 \xrightarrow{i_1} X_1 \amalg X_2 \xleftarrow{i_2} X_2 \qquad (*)$$

$$x \mapsto (x,1) \qquad (x,2) \mapsfrom x$$

- any map $f: X_1 \amalg X_2 \to Y$ is completely determined by restrictions to X_1, X_2, i.e. by $f_1 = f \circ i_1$, $f_2 = f \circ i_2$.

$$\begin{tikzcd} X_1 \arrow[d, "i_1"'] \arrow[dr, "f_1"] & \\ X_1 \amalg X_2 \arrow[r, dashed, "\exists! f"'] & Y \\ X_2 \arrow[u, "i_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

def Let X_1, X_2 be objects in a category $\mathcal{C}$. $X \in Ob(\mathcal{C})$ is called a coproduct of X_1, X_2 (notation: $X_1 \amalg X_2$) if there are morphisms $X_1 \xrightarrow{i_1} X \xleftarrow{i_2} X_2$ s.t. this pair of maps satisfies the univ. property expressed by the comm diag

$$\begin{tikzcd} X_1 \arrow[d, "i_1"'] \arrow[dr, "f_1"] & \\ X \arrow[r, dashed, "\exists! f"'] & Y \\ X_2 \arrow[u, "i_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

coproducts may fail to exist.
Ex: $C = \{$ sets with 3-elements + + maps between them $\}$

Ex: (Coproducts in some categories)

$\mathcal{C}$	coproduct	
Set	$X_1 \amalg X_2$	disj. union
Vect	$X_1 \oplus X_2$	direct sum
Grp	$X_1 * X_2$	free product
Top	$X_1 \amalg X_2$	disj. union
Top_*	$X_1 \vee X_2$	wedge sum (or wedge product)

$$X_1 \vee X_2 = X_1 \amalg X_2 / \{i_1(x_1), i_2(x_2)\}$$

Pushouts

motivating example - in Top

let $X_1, X_2 \subset X$ (open) ← top. sp., $X = X_1 \cup X_2$, then we have the comm. square

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, "i_1"] \arrow[d, "i_2"'] & X_1 \arrow[d, "k_1"] \\ X_2 \arrow[r, "k_2"'] & X \end{tikzcd}$$

let $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ cont. maps which agree on $X_1 \cap X_2$

Then $\exists!$ well-defined cont. map $f : X \to Y$ s.t. $f|_{X_i} = f_i$ (f is "glued" out of maps f_1, f_2)

def In a cat. $\mathcal{C}$, a comm. diagram of objects & morphisms

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[d, "j_2"'] & X_1 \arrow[d, "k_1"] \\ X_2 \arrow[r, "k_2"'] & X \end{tikzcd}$$

is a "pushout diagram" if it satisfies the univ. property expressed by

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[d, "j_2"'] & X_1 \arrow[d, "k_1"] \arrow[ddr, bend left, "f_1"] & \\ X_2 \arrow[r, "k_2"'] \arrow[drr, bend right, "f_2"'] & X \arrow[dr, dashed, "\exists! f"] & \\ & & Y \end{tikzcd}$$

The object X is called the pushout of the diagram

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[d, "j_2"'] & X_1 \\ X_2 & \end{tikzcd}$$

Ex:

Category	pushout
Set	$X_1 \cup_A X_2$
Top	$X_1 \cup_A X_2$
Top$_*$	$X_1 \cup_A X_2$
Grp	$X_1 *_A X_2$
Vect	$X_1 \oplus X_2 / (j_1 - j_2)(A)$

$:= X_1 \amalg X_2 / i_1(j_1(a)) \sim i_2(j_2(a))$ from $(*)$

amalgamated free product

Seifert-van Kampen: functor $\pi_1 : Top_* \to Grp$ "preserves pushouts":

$$\underbrace{\begin{tikzcd} A \arrow[r, "j_1"] \arrow[d, "j_2"'] & X_1 \arrow[d, "k_1"] \\ X_2 \arrow[r, "k_2"'] & X \end{tikzcd}}_{\text{pushout diagram in } Top_*} \quad \overset{\pi_1}{\Longrightarrow} \quad \underbrace{\begin{tikzcd} \pi_1(A) \arrow[r, "(j_1)_*"] \arrow[d, "(j_2)_*"'] & \pi_1(X_1) \arrow[d, "(k_1)_*"] \\ \pi_1(X_2) \arrow[r, "(k_2)_*"'] & \pi_1(X) \end{tikzcd}}_{\text{pushout in } Grp}$$

(Assuming A connected)

Covering spaces

def A map $p : \tilde{X} \to X$ is a covering map if $\forall x \in X$ $\exists$ an open nbhd $U \subset X$ ($U \ni x$) s.t. $p^{-1}(U) = \coprod_i U_i \subset \tilde{X}$ s.t. $p|_{U_i} : U_i \to U$ is a homeomorphism for each U_i.

Such U is called evenly covered. $\tilde{X}$ is called a covering space for X.

Prop[a] <"lemma (a)"> (Unique path lifting for covering spaces)

Let $\gamma : I \to X$ a path starting at $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$.
Then $\exists!$ path $\tilde{\gamma} : I \to \tilde{X}$ s.t.
- $p \circ \tilde{\gamma} = \gamma$ and
- $\tilde{\gamma}(0) = \tilde{x}_0$.

- $\tilde{\gamma}$ is called the "lift" of γ.

(Proof: as lemma (a))

Prop[c] <"lemma (c)"> (Unique homotopy lifting)

Let $p : \tilde{X} \to X$ a covering space, $f_t : Y \to X$ a homotopy (of maps from Y) and $\tilde{f}_0 : Y \to \tilde{X}$ a map lifting f_0. Then $\exists!$ homotopy $\tilde{f}_t$ starting with $\tilde{f}_0$ that lifts f_t.

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}_0"] \arrow[d, "i"'] & \tilde{X} \arrow[d, "p"] \\ Y \times I \arrow[r, "f"'] \arrow[ur, dashed, "\exists! \tilde{f}"] & X \end{tikzcd}$$

(Proof: as lemma (c))

- Case $Y = I$: $f_0 = $ path in X $\xrightarrow{\text{lift}}$ $\tilde{f}_0 = $ path in $\tilde{X}$

if f_t - homotopy of paths in X $\Rightarrow$ $\tilde{f}_t$ - homotopy of paths in $\tilde{X}$

<u>Observation</u>: homotopy $\tilde{f}_t$ preserves endpoints iff f_t preserves endpoint

$\lceil$ $\Rightarrow$) obvious: $\underset{\text{or } 1}{f_t(0)} = p(\underset{\text{or } 1}{\tilde{f}_t(0)})$ - constant

$\Leftarrow$ $\underset{\text{or } 1}{f_t(0)}$, $t \in [0,1]$ - constant path in X $\longrightarrow$ $\tilde{f}_t(0)$, $t \in [0,1]$ some path in $\tilde{X}$, but must be constant by uniqueness of path lifting $\tilde{f}_t(0) = \tilde{f}_0(0)$ $\rfloor$

<u>Proposition</u>:

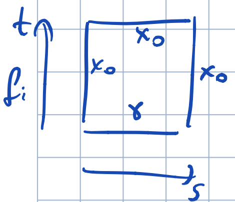
Let $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a covering map. Then

(i) The induced homomorphism $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is <u>injective</u>.

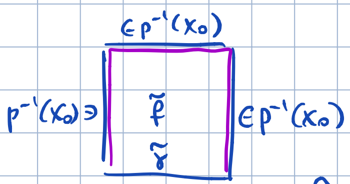
(ii) A based loop γ in (X, x_0) represents an element of the image of p_* iff its (unique) lift $\tilde{\gamma}: I \longrightarrow \tilde{X}$ with $\tilde{\gamma}(0) = \tilde{x}_0$ is a <u>loop</u>, i.e., $\tilde{\gamma}(1) = \tilde{x}_0$.

Proof Use homotopy lifting property, for $Y = I$.

(i) let $\tilde{\gamma}$ be a based loop in $\ker p_*$ in $(\tilde{X}, \tilde{x}_0)$. Let $f: \underset{s}{I} \times \underset{t}{I} \longrightarrow X$ be a homotopy from $\overset{\text{"}\gamma}{p \circ \tilde{\gamma}}$ to $\underset{\substack{\text{const}\\\text{loop}}}{C_{x_0}}$



Using Prop (c): let $\tilde{f}$ be a lift of f with $f|_{t=0} = \tilde{\gamma}$



by uniqueness of path lifting, $\tilde{f}$ on left-top-right sides is the const. path $C_{\tilde{x}_0}$.

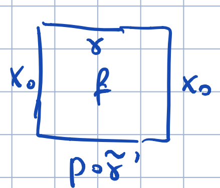
~~$\Rightarrow \tilde{\gamma}(1) = \tilde{\gamma}(0) = x_0 \Rightarrow \tilde{\gamma}$ a loop based at $\tilde{x}_0$.~~

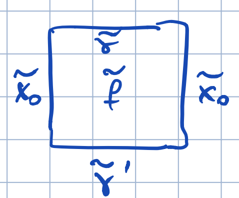
$\tilde{f}$ is a homotopy between $\tilde{\gamma}$ and $C_{\tilde{x}_0}$

$\Rightarrow [\tilde{\gamma}] = 0$.

(ii) let $\underset{\substack{\uparrow\\\text{based loop in } X, x_0}}{[\gamma]} \in \operatorname{im} p_*$. I.e. $\exists\ \underset{\substack{\text{based}\\\text{Loop in } \tilde{X}, \tilde{x}_0}}{\tilde{\gamma}'}$ s.t. $[\gamma] = p_*[\tilde{\gamma}'] = [p \circ \tilde{\gamma}']$

let f - homotopy between γ and $p \circ \tilde{\gamma}'$, and $\tilde{f}$ its lift with $f|_{t=0} = \tilde{\gamma}'$.





$\Rightarrow$ the lift $\tilde{\gamma}$ of γ starting at $\tilde{x}_0$ also ends at $\tilde{x}_0$.

$\Rightarrow \tilde{\gamma}$ is a based <u>loop</u>.

$\boxed{/}$