Organizational: Sep ~~25~~ 21, 1pm — Sep ~~25~~ 23, 11 am (take-home) - midterm exam.

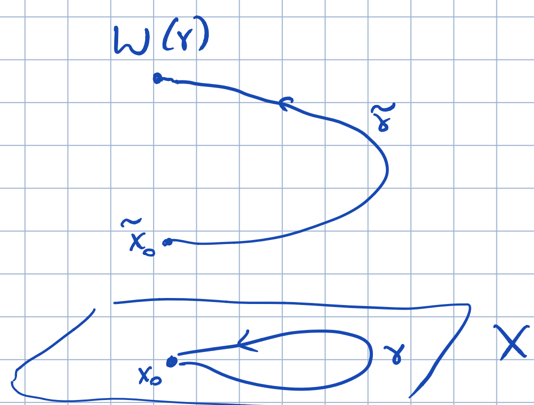
Proposition:

Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map. Then

(i) The induced homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

(ii) A based loop γ in (X, x_0) represents an element of the image of p_* iff its (unique) lift $\tilde{\gamma} : I \to \tilde{X}$ with $\tilde{\gamma}(0) = \tilde{x}_0$ is a loop, i.e., $\tilde{\gamma}(1) = \tilde{x}_0$.

- Rem: for a covering $p : \tilde{X} \to X$, one does not have to include surjectivity of p as an axiom. For X path-connected, it follows from existence of path-lifting:
($\forall x \in X$ take some path γ from x_0 to x → it lifts to $\tilde{X}$ $\Rightarrow p(\tilde{\gamma}(1)) = x$
$\Rightarrow \mathrm{Im}(p) = X$)



- for $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ covering map, introduce the map

$W : \{\text{based loops in } (X, x_0)\} \longrightarrow p^{-1}\{x_0\}$

$\gamma \longmapsto \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the lifting of γ with $\tilde{\gamma}(0) = \tilde{x}_0$.

(ω generalizes the idea of a winding number $\omega : \{\text{loops in } S^1\} \to \mathbb{Z}$ in the case of $p: \mathbb{R} \to S^1$)

Proposition Let $\tilde{X}$ be path-connected and let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map. Let $G = \pi_1(X, x_0)$ and $H := p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset G$ (subgroup).

Let $H \backslash G = \{Hg \mid g \in G\} = G / g \sim hg \ \forall g \in G, h \in H$

– the set of left H-cosets.

Then the map $\Psi : H \backslash G \longrightarrow p^{-1}(x_0)$, $[\gamma] \longmapsto \omega(\gamma)$ is a well-defined bijection.

In particular, the number of sheets of $\tilde{X} \to X$ (the cardinality of $p^{-1}(x_0)$) is equal to the index $[G : H]$ of the subgroup $H \subset G$.

$:= \# H \backslash G$

Proof

- well-definedness: $g = [\gamma] \overset{\Psi}{\longmapsto} \omega(\gamma)$

$hg = [\delta] \cdot [\gamma] = [p \circ (\tilde{\delta} * \tilde{\gamma})] \overset{\Psi}{\longmapsto} \omega(\gamma) = \Psi(g)$ ✓

$[\delta]$ lifts to a loop $\tilde{\delta}$; $[\gamma]$ lifts to a path $\tilde{\gamma}$; $\tilde{\delta} * \tilde{\gamma}$ starts at $\tilde{x}_0$, ends at $\omega(\gamma)$

- Surjectivity: (use that $\tilde{X}$ is path connected). For $\tilde{x}_1 \in p^{-1}(x_0)$, choose any path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then $\Psi([p \circ \tilde{\gamma}]) = \tilde{\gamma}(1) = \tilde{x}_1$. ✓ ($[p \circ \tilde{\gamma}]$: loop in X)

- injectivity: let $g_1, g_2 \in \pi_1(X, x_0)$ with $\Psi(g_1) = \Psi(g_2) =: \tilde{x}_1$

$g_1 = [\gamma_1]$, $g_2 = [\gamma_2]$, $\Psi(g_1) = \omega(\gamma_1)$, $\Psi(g_2) = \omega(\gamma_2)$

$\tilde{\gamma}_1 * \overline{\tilde{\gamma}_2}$ – based loop in $(\tilde{X}, \tilde{x}_0)$ $\Rightarrow$ $H \ni [p(\tilde{\gamma}_1 * \overline{\tilde{\gamma}_2})] = g_1 g_2^{-1}$

($\tilde{\gamma}_1$ from $\tilde{x}_0$ to $\tilde{x}_1$; $\overline{\tilde{\gamma}_2}$ from $\tilde{x}_1$ to $\tilde{x}_0$)

$\Rightarrow g_1 = h g_2$ for some $h \in H$ $\overset{\text{left } H\text{-cosets}}{\Rightarrow}$ $Hg_1 = Hg_2$. ✓

□

Lifting maps Given a covering $\begin{matrix}(\tilde{X},\tilde{x}_0)\\ \downarrow p\\ (X,x_0)\end{matrix}$ and a map $(Y,y_0)\xrightarrow{f}(X,x_0)$,

we are interested in a lifting

$$\begin{array}{ccc} & & (\tilde{X},\tilde{x}_0) \\ & \overset{\tilde{f}}{\nearrow} & \downarrow p \\ (Y,y_0) & \xrightarrow[f]{} & (X,x_0) \end{array} \qquad (*)$$

necessary condition for existence of $\tilde{f}$: $f_*\pi_1(Y,y_0)\subset p_*\pi_1(\tilde{X},x_0)$, since we have

$$\begin{array}{ccc} & & \pi_1(\tilde{X},\tilde{x}_0) \\ & \overset{\tilde{f}_*}{\nearrow} & \downarrow p_* \text{ - inclusion!} \\ \pi_1(Y,y_0) & \xrightarrow[f_*]{} & \pi_1(X,x_0) \end{array}$$

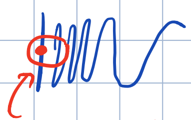
This is also a sufficient condition if Y is not "too wild".

def A top space Y is "locally path-connected" if $\forall y\in Y$ and any nbhd U of y, there is an open nbhd $V\subset U$ which is path-connected.

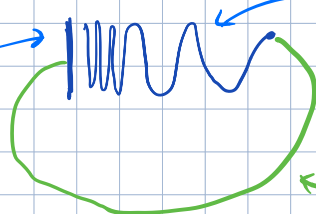
More generally, if P is some property of a top space (e.g. compact, connected, ...), then Y is "locally P" if $\forall y\in Y$, $\forall U \ni y$ nbhd, $\exists V\subset U$ open s.t. V has property P.

Ex:

- $Y=Y_1 \sqcup Y_2$ is locally path connected but not path connected (Y_1, Y_2 path connected manifolds)

- topologist's sine curve  connected but not locally path connected.

- the "Warsaw circle"



- path connected but not locally path connected

Proposition (lifting criterion) Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a covering map and $f:(Y,y_0)\to(X,x_0)$ a (basepoint preserving) map whose domain Y is path connected and locally path connected. Then a lift $\tilde{f}$ in $(*)$ exists iff $f_*\pi_1(Y,y_0)\subset p_*\pi_1(\tilde{X},\tilde{x}_0)$. There is at most one such lift.

Rem The hypothesis that Y is LPC cannot be dropped.

Ex: $Y =$ Warsaw circle, $\pi_1(Y) = 0$

$f : Y \to S^1$ wrapping Y around S^1 once — this map does not have a lift $\tilde{f} : Y \to \mathbb{R}$.

Proof of lifting criterion: $\Rightarrow$ (lift $\exists \Rightarrow f_* \pi_1(Y) \subset \pi_1(\tilde{X})$) – already proved.

uniqueness: let $\tilde{f}, \tilde{f}'$ two lifts, $y \in Y$ and γ a path from y_0 to y ($I \to Y$)

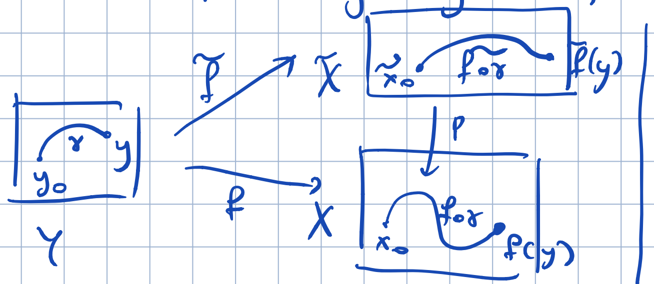
then $\tilde{f}\circ\gamma$, $\tilde{f}'\circ\gamma$ two paths in $\tilde{X}$ starting at $\tilde{x}_0$ and projecting to $f\circ\gamma$ in X

$\Rightarrow$ by uniqueness of lifted paths, $\underbrace{\tilde{f}\circ\gamma(1)}_{\tilde{f}(y)} = \underbrace{\tilde{f}'\circ\gamma(1)}_{\tilde{f}'(y)}$. This is so for any $y \in Y$ $\Rightarrow \tilde{f} = \tilde{f}'$ ✓

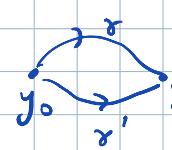
$\Leftarrow$ (Construction of the lift $\tilde{f} : Y \to \tilde{X}$)

$\tilde{f} : y \longmapsto \widetilde{f\circ\gamma}(1)$ — lift of $f\circ\gamma$ to a path in $\tilde{X}$ starting at $\tilde{x}_0$

where γ – path from y_0 to y in Y,



• $\tilde{f}$ is well-defined: if



two paths,

$\gamma * \bar{\gamma}'$ – based loop in Y

$\Rightarrow f\circ(\gamma * \bar{\gamma}')$ – based loop in X

$[f\circ(\gamma * \bar{\gamma}')] = [p \circ \delta]$ ← δ based loop in $\tilde{X}$

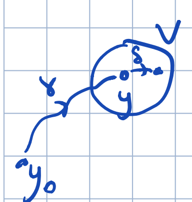
$f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0)$

by uniqueness of lifted paths:

$\delta = \widetilde{f\circ\gamma} * \widetilde{f\circ\bar{\gamma}'}$ $\Rightarrow$ endpoints ($t=1$) of $\widetilde{f\circ\gamma}$ and $\widetilde{f\circ\gamma'}$ coincide! ✓

↑ based loop

Continuity of $\tilde{f}$ – check in a nbhd of $y \in Y$. Use LPC: take $V \subset Y$ ($y \in V$) open PC subnbhd of $f^{-1}(U)$ ← evenly covered in X.



$\tilde{f}(y') = \widetilde{f(\gamma * \delta)}(1) = p_i^{-1} f(y')$ – depends continuously on y'! ✓

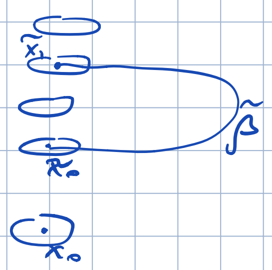
$y' \in V$

$p_i : \tilde{U}_i \xrightarrow{\cong} U$

($\tilde{U}_i$) contains $\tilde{f}(y)$

☑

Lemma Let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map and $\tilde{x}_1 \in p^{-1}(x_0)$.



Let $\tilde{\beta}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$ in $\tilde{X}$ and $b = [p \circ \tilde{\beta}] \in \pi_1(X, x_0)$.

Let $H = p_* \pi_1(\tilde{X}, \tilde{x}_0)$. Then

$$p_* \pi_1(\tilde{X}, \tilde{x}_1) = b^{-1} H b \quad \subset \pi_1(X, x_0)$$

– conjugate subgroup.

Proof:

$$\Phi_\beta : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(\tilde{X}, \tilde{x}_1) \quad \text{– isomorphism (change of base point)}$$

$$[\tilde{\gamma}] \longmapsto [\bar{\tilde{\beta}} * \tilde{\gamma} * \tilde{\beta}]$$

$\downarrow p_*$

$\pi_1(X, x_0)$

$$b^{-1} \cdot [p(\tilde{\gamma})] \cdot b$$

□

Classification of coverings

def A covering $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is called a <u>universal covering</u> if $\tilde{X}$ is simply connected. In that case, $\tilde{X}$ is called the <u>universal covering space</u> of X.

- If X is PC and LPC, a univ. covering $\tilde{X}$ satisfies the univ. property:

(*)
$$(\tilde{X}, \tilde{x}_0) \overset{\exists!\, f}{\dashrightarrow} (\tilde{X}', \tilde{x}_0')$$
$$p \searrow \quad \swarrow p'$$
$$(X, x_0)$$

for any covering $p': (X', x_0') \to (X, x_0)$

– by Lifting Criterion, with $Y = \tilde{X}$:

- $\tilde{X}$ simply conn → PC, → $p_* \pi_1(\tilde{X}) = 0 \subset p'_* \pi_1(\tilde{X}')$
- X LPC ⇒ $\tilde{X}$ LPC

} ⇒ Criterion applies

$\tilde{U}_i$, $\tilde{x}'$, x, U

evenly covered for p and p'

$f|_{\tilde{U}_j} = (p'_i)^{-1} \circ p$

– f is itself a covering ⟨normally, one would also assume $\tilde{X}'$ is PC⟩

- Univ. covering $\tilde{X}$ of a PC, LPC space X is unique up to isomorphism – by (*)

⇒ one speaks of "the" univ. covering of X.

Ex: 1) $p: \mathbb{R} \to S^1$ – univ. covering of S^1, since $\mathbb{R}$ simply connected
$\quad t \longmapsto e^{2\pi i t}$

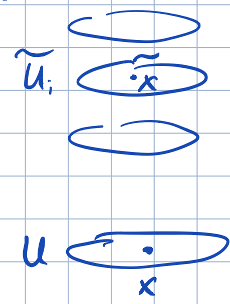
2) $p: S^n \to \mathbb{R}P^n = S^n/\sim$ – univ. covering of $\mathbb{R}P^n$, since S^n simply connected
$\quad n \geq 2$

(SSC)

def A space X is semilocally simply connected if $\forall x \in X$ has a open nbhd U s.t. the induced homomorphism $\pi_1(U,x) \longrightarrow \pi_1(X,x)$ is the trivial map.

- SSC is a necessary condition for the existence of a univ. covering $\tilde{X} \to X$ for X path connected. Indeed: for $x \in X$, let $U \subset X$ be an evenly covered open nbhd,

$\tilde{U}_i \xrightarrow[p]{\approx} U$, $\tilde{x}$ - preimage of x in $\tilde{U}_i$.



Let $i: U \hookrightarrow X$, $\tilde{i}: \tilde{U}_i \hookrightarrow \tilde{X}$ inclusions

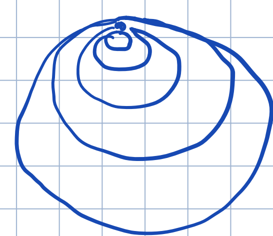
We have:

$$\begin{array}{ccc} \pi_1(\tilde{U}_i, \tilde{x}) & \xrightarrow{(\tilde{i})_*} & \pi_1(\tilde{X}, \tilde{x}) = 1 \\ p_* \downarrow \cong & & \downarrow p_* \\ \pi_1(U, x) & \xrightarrow[i_*]{} & \pi_1(X, x) \end{array}$$

$\Rightarrow$ homomorphism i_* must be trivial!

Ex: An example of a space which is not SSC: "Hawaiian earring"

$X = \bigcup_{n \geq 1}$ circle of radius $1/n$ in $\mathbb{R}^2$ with center $(0, -\frac{1}{n})$ $\subset \mathbb{R}^2$



- "locally simply-connected" is a stronger property than "semi-locally simply-connected".

E.g. - Warsaw circle is not LPC (⇒ not LSC) but has $\pi_1 = 0 \Rightarrow$ SSC

- Cone(Hawaiian Earring) has $\pi_1 = 0$ (⇒ SSC) but not every point has a simply-con subnbhd of any prescribed nbhd.

Theorem

A path connected space X has a univ. covering iff X is semilocally simply-connected.

<proof ⇒ - saw above ⇐ - in Hatcher or Munkres>