Lifting maps Given a covering
$$\begin{tikzcd} (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (X, x_0) \end{tikzcd}$$
and a map $(Y, y_0) \xrightarrow{f} (X, x_0)$,

we are interested in a *lifting*
$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd} \qquad (*)$$

necessary condition for existence of $\tilde{f}$: $f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, x_0)$, since we have
$$\begin{tikzcd} & \pi_1(\tilde{X}, \tilde{x}_0) \arrow[d, "p_* \text{ - inclusion!}"] \\ \pi_1(Y, y_0) \arrow[ur, dashed, "\tilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(X, x_0) \end{tikzcd}$$
.

This is also a sufficient condition if Y is not "too wild".

Proposition (lifting criterion) Let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map and $f: (Y, y_0) \to (X, x_0)$ a (basepoint preserving) map, whose domain Y is path connected and locally path connected. Then a lift $\tilde{f}$ in $(*)$ exists iff $f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0)$. There is at most one such lift.

Rem The hypothesis that Y is LPC (path connected locally) cannot be dropped.

Ex: $Y =$ Warsaw circle, $\pi_1(Y) = 0$

$f: Y \to S^1$ wrapping Y around S^1 once — this map does not have a lift $\tilde{f}: Y \to \mathbb{R}$.

Proof of lifting criterion: $\Rightarrow$ (lift $\exists \Rightarrow f_*\pi_1(Y) \subset \pi_1(\tilde{X})$) — already proved.

uniqueness: let $\tilde{f}, \tilde{f}'$ two lifts, $y \in Y$ and γ a path from y_0 to y ($\gamma: I \to Y$)

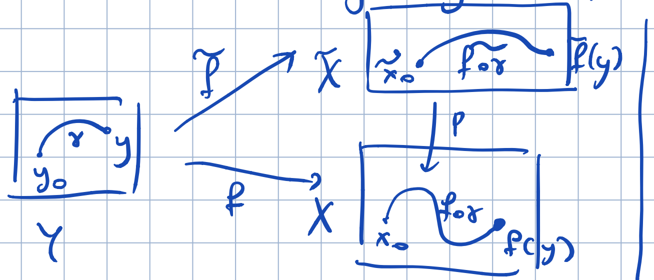
then $\tilde{f}\circ\gamma$, $\tilde{f}'\circ\gamma$ two paths in $\tilde{X}$ starting at $\tilde{x}_0$ and projecting to $f\circ\gamma$ in X

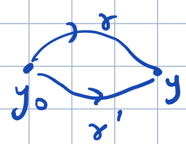
$\Rightarrow$ by uniqueness of lifted paths, $\underbrace{\tilde{f}\circ\gamma(1)}_{\tilde{f}(y)} = \underbrace{\tilde{f}'\circ\gamma(1)}_{\tilde{f}'(y)}$. This is so for any $y \in Y$ $\Rightarrow \tilde{f} = \tilde{f}'$ ✓

$\Leftarrow$ (Construction of the lift $\tilde{f}: Y \to \tilde{X}$)

$\tilde{f}: y \longmapsto \widetilde{f\circ\gamma}(1)$ — lift of $f\circ\gamma$ to a path in $\tilde{X}$ starting at $\tilde{x}_0$

where γ - path from y_0 to y in Y,



• $\tilde{f}$ is well-defined: if  two paths,

$\gamma * \bar{\gamma}'$ - based loop in Y

$\Rightarrow f\circ(\gamma * \bar{\gamma}')$ - based loop in X

$f\circ(\gamma*\bar{\gamma}') = p\circ\tilde{\delta}$ — based loop in $\tilde{X}$

since $f_*\pi_1(Y, y_0) \subset p_*\pi_1(\tilde{X}, \tilde{x}_0)$

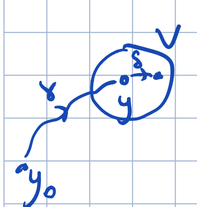
by uniqueness of lifted paths:

$\tilde{\delta} = \widetilde{f\circ\gamma} * \widetilde{f\circ\bar{\gamma}'}$ $\Rightarrow$ endpoints ($t=1$) of $\widetilde{f\circ\gamma}$ and $\widetilde{f\circ\gamma'}$ coincide! ✓

based loop

Continuity of $\tilde{f}$ — check in a nbhd of $y \in Y$. Use LPC: take $V \subset Y$ ($V \ni y$) open PC nbhd s.th. of $f^{-1}(U)$

evenly covered in X.



$\tilde{f}(y') = \widetilde{f(\gamma*\delta)}(1) = p_i^{-1} f(y')$ — depends continuously on y'! ✓

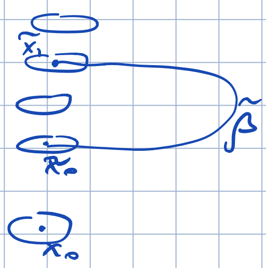
$y' \in V$

$p_i: \tilde{U}_i \xrightarrow{\cong} U$

contains $\tilde{f}(y)$

$\square$

Lemma Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a covering map and $\tilde{x}_1\in p^{-1}(x_0)$. Let $\tilde{\beta}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$ in $\tilde{X}$ and $b=[p\circ\tilde{\beta}]\in\pi_1(X,x_0)$. Let $H=p_*\pi_1(\tilde{X},\tilde{x}_0)$. Then



$$p_*\pi_1(\tilde{X},\tilde{x}_1)=b^{-1}Hb\quad\subset\pi_1(X,x_0)$$

– conjugate subgroup.

Proof:

$$\begin{aligned}\Phi_\beta:\pi_1(\tilde{X},\tilde{x}_0)&\to\pi_1(\tilde{X},\tilde{x}_1)\\ [\tilde{\gamma}]&\mapsto[\bar{\tilde{\beta}}*\tilde{\gamma}*\tilde{\beta}]\end{aligned}$$

– isomorphism (change of base point)

$$[\bar{\tilde{\beta}}*\tilde{\gamma}*\tilde{\beta}]\xrightarrow{p_*}\pi_1(X,x_0)$$

$$[\tilde{\gamma}]\mapsto b^{-1}\cdot[p(\tilde{\gamma})]\cdot b$$

$\square$

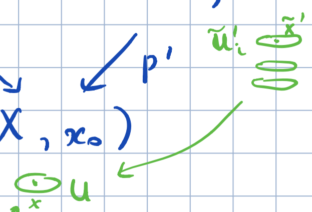
Classification of coverings

def A covering $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ is called a universal covering if $\tilde{X}$ is simply connected. In that case, $\tilde{X}$ is called the universal covering space of X.

• If X is PC and LPC, a univ. covering $\tilde{X}$ satisfies the univ. property:

$$\begin{array}{ccc}(\tilde{X},\tilde{x}_0) & \overset{\exists!\, f}{\dashrightarrow} & (\tilde{X}',\tilde{x}_0')\\ & p\searrow \quad \swarrow p' & \\ & (X,x_0) & \end{array}\qquad (*)$$

for any covering $p':(X',x_0')\to(X,x_0)$



evenly covered for p and p'

$f|_{\tilde{U}_j}=(p_i')^{-1}\circ p$

– by Lifting Criterion, with $Y=\tilde{X}$:

. $\tilde{X}$ simply conn $\to$ PC, $\to p_*\pi_1(\tilde{X})=0\subset p'_*\pi_1(\tilde{X}')$ $\Big\}$ $\Rightarrow$ Criterion applies

. X LPC $\Rightarrow\tilde{X}$ LPC

– f is itself a covering

<normally, one would also assume $\tilde{X}'$ is PC>

• Univ. covering $\tilde{X}$ of a PC, LPC space X is unique up to isomorphism – by $(*)$

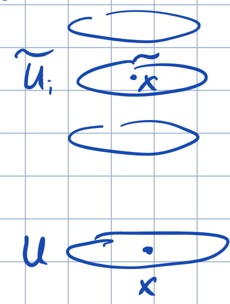
$\Rightarrow$ one speaks of "the" univ. covering of X.

Ex: 1) $p:\mathbb{R}\to S^1$, $t\mapsto e^{2\pi i t}$ – univ. covering of S^1, since $\mathbb{R}$ simply connected

2) $p:S^n\to\mathbb{R}P^n=S^n/\sim$, $n\geq 2$ – univ. covering of $\mathbb{R}P^n$, since S^n simply connected

def A space X is semilocally simply connected (SSC) if $\forall x \in X$ has a open nbhd U s.t. the induced homomorphism $\pi_1(U,x) \longrightarrow \pi_1(X,x)$ is the trivial map.

- SSC is a necessary condition for the existence of a univ. covering $\tilde{X} \to X$ for X path connected. Indeed: for $x \in X$, let $U \subset X$ be an evenly covered open nbhd, $\tilde{U}_i \xrightarrow[p]{\cong} U$, $\tilde{x}$ - preimage of x in $\tilde{U}_i$.



Let $\iota: U \hookrightarrow X$, $\tilde{\iota}: \tilde{U}_i \hookrightarrow \tilde{X}$ inclusions

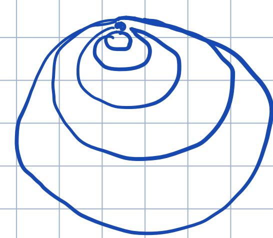
We have:

$$\begin{tikzcd}
\pi_1(\tilde{U}_i, \tilde{x}) \arrow[r, "(\tilde{\iota})_*"] \arrow[d, "p_*"', "\cong"] & \pi_1(\tilde{X}, \tilde{x}) = 1 \arrow[d, "p_*"] \\
\pi_1(U, x) \arrow[r, "\iota_*"'] & \pi_1(X, x)
\end{tikzcd}$$

$\Rightarrow$ homomorphism ι_* must be trivial!

Ex: An example of a space which is not SSC: "Hawaiian earring"

$X = \bigcup_{n \geq 1}$ circle of radius $1/n$ in $\mathbb{R}^2$ with center $(0, -\frac{1}{n})$ $\subset \mathbb{R}^2$



- "locally simply-connected" is a stronger property than "semi-locally simply-connected".

E.g. - Warsaw circle is not LPC (⇒ not LSC) but has $\pi_1 = 0 \Rightarrow$ SSC

- Cone(Hawaiian Earring) has $\pi_1 = 0$ (⇒ SSC) but not every point has a simply-con subnbhd of any prescribed nbhd.

def A top. space X is "reasonable" if it is locally path-connected and semilocally simply connected (LPC+SSC)

Theorem

A reasonable path connected space X has a univ. covering $\begin{array}{c}\tilde{X}\\ \downarrow p \\ X\end{array}$.

(Munkres, Thm 82.1; Hatcher pp. 64-65)

• Construction of the univ. covering for a path-connected reasonable X:

fix $x_0 \in X$. $\tilde{X} := \{\text{paths in } X \text{ starting at } x_0\} / \text{homotopy of paths}$

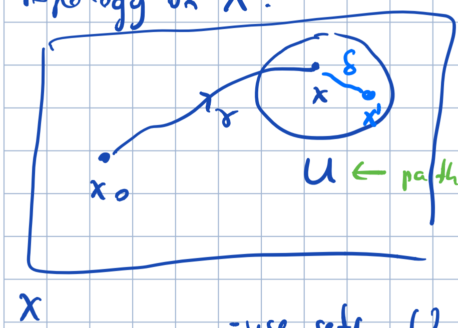
$\downarrow p: \gamma \mapsto \gamma(1)$

X

motivation: $\tilde{X}$ simply conn. $\Rightarrow \exists$ unique homotopy class of paths from $\tilde{x}_0$ to any $\tilde{x}$

- lifting of a path (up to homotopy) in X starting at x_0

topology on $\tilde{X}$:



$U \leftarrow$ path-conn. nbhd s.t. $\pi_1(U) \xrightarrow{i_*} \pi_1(X)$ is trivial

$U_{[\gamma]} := \{[\gamma * \delta] \mid \delta - \text{path in } U \text{ starting at } x\}$

$p: U_{[\gamma]} \to U$ - surjective - since U path-conn

- injective - since two paths in U x to x' are homotopic in X by

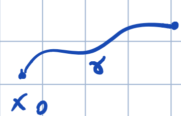
- use sets $U_{[\gamma]}$ as a basis for topology on $\tilde{X}$. [Hatcher, p. 64]

$\langle$ {all path-conn U's with $\pi_1(U) \xrightarrow[i_*]{} \pi_1(X)$ triv.} form a basis for topology on $X \rangle$

- U is evenly covered by $\coprod_{[\gamma]} U_{[\gamma]}$ $\quad$ $[\gamma] \leftarrow$ homotopy classes of paths $x_0 \to x$

- $\tilde{X}$ is path-conn and $\pi_1(\tilde{X}) = 1$

• $\tilde{X}$ path-conn?: $[\gamma] \in \tilde{X} \longrightarrow$ path $\Gamma_t = [\gamma(ts): \overset{s}{I} \to X]$ $\quad (*)$



$\Gamma_1 = [\gamma]$, $\Gamma_0 = C_{x_0}$

so: Γ is a path from C_{x_0} to $[\gamma]$ ✓

arbitrary pt of $\tilde{X}$

• $\pi_1(\tilde{X}) = 1$? $\iff p_* \pi_1(\tilde{X}) \overset{!}{=} 1 \subset \pi_1(X)$

Let γ based loop in X, x_0 lifting to a loop in $(\tilde{X}, C_{x_0})$

path $t \mapsto [\Gamma_t]$ is the lift of γ to $\tilde{X}$ starting at C_{x_0}.

it is a loop $\overset{(*)}{\Longrightarrow} [\Gamma_1] = [\Gamma_0]$ $\quad$ which proves ✓.

$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{[\gamma]} \quad\quad \overset{\shortparallel}{[C_{x_0}]}$

Classification theorem (for covering spaces)

(A) Let X be path-connected, reasonable. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-conn. coverings $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups $H \subset \pi_1(X, x_0)$, obtained as $H = p_* \pi_1(\tilde{X}, \tilde{x}_0)$.

(B) if basepoints are ignored, this correspondence gives a bijection

$p: \tilde{X} \to X /_{\text{iso}} \quad \xleftrightarrow{1-1} \quad$ conjugacy classes of subgroups in $\pi_1(X, x_0)$

- Group actions

def Let G be a topological group.

(i) A G-action on a space X is a cont. map $\begin{array}{c} G\times X \xrightarrow{\mu} X \\ g, x \longmapsto gx \end{array}$ satisfying $g(hx) = (gh)x$ for $g,h \in G$, $x\in X$.

- If G is a discrete group, continuity of μ is equiv. to continuity of

$$\begin{array}{c} \mu(g,-): X \longrightarrow X \\ x \longmapsto gx \end{array}, \quad \forall g \in G$$

(ii) The action is free if $\forall x \in X$, $gx = x \iff g = \underset{\uparrow\text{ unit in } G}{1}$

(iii) Action is transitive if $\forall x,y \in X$ $\exists g\in G$ s.t. $g\cdot x = y$

(iv) For $x\in X$, the subset $Gx := \{gx \mid g\in G\} \subset X$ is the "orbit through x."

$G\backslash X = \{\text{orbits}\}$ is the orbit space of the G-action on X.

Topology on $G\backslash X$ - the quotient topology determined by $\begin{array}{c} p: X \longrightarrow G\backslash X \\ x \longmapsto Gx \end{array}$

Examples of group actions

(1) $\mathbb{Z}$ acts on $\mathbb{R}$ via $(n,t) \longmapsto n+t$. Orbit space $\begin{array}{c} \mathbb{Z}\backslash\mathbb{R} \approx S^1 \\ [t] \mapsto e^{2\pi i t} \end{array}$

Thus, the proj. map $\begin{array}{c} \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \xrightarrow{\approx} S^1 \\ t \longmapsto e^{2\pi i t} \end{array}$ is our stand. covering of S^1.

(2) $\mathbb{Z}_2 = \{\pm 1\}$ acts on S^n via $\begin{array}{c} \{\pm1\} \times S^n \longrightarrow S^n \\ (\pm 1, x) \longmapsto \pm x \end{array}$. Orbit space: $\{\pm1\}\backslash S^n = \mathbb{R}P^2$.

- in these examples, $p: X \to G\backslash X$ are the univ. coverings for the quotient S^1 or $\mathbb{R}P^2$.

G is π_1 of the quotient.

Lemma: Let $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be the univ. covering of a path-conn, LR space X.
Then $\underset{G=}{\pi_1(X,x_0)}$ acts freely on $\tilde{X}$ so that G-orbits are the fibers of p.