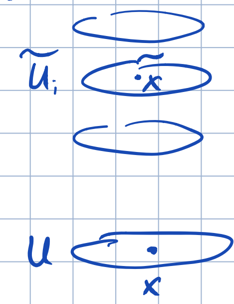
def A space X is semilocally simply connected (SSC) if $\forall x \in X$ has a open nbhd U s.t. the induced homomorphism $\pi_1(U,x) \longrightarrow \pi_1(X,x)$ is the trivial map.

- SSC is a necessary condition for the existence of a univ. covering $\tilde{X} \to X$ for X path connected. Indeed: for $x \in X$, let $U \subset X$ be an evenly covered open nbhd,

$\tilde{U}_i \xrightarrow[p]{\approx} U$, $\tilde{x}$ - preimage of x in $\tilde{U}_i$.



Let $\iota : U \hookrightarrow X$, $\tilde{\iota} : \tilde{U}_i \hookrightarrow \tilde{X}$ inclusions

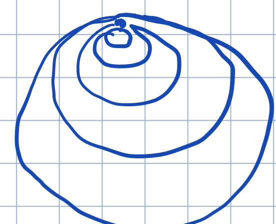
We have:

$$\begin{tikzcd}
\pi_1(\tilde{U}_i, \tilde{x}) \arrow[r, "(\tilde{\iota})_*"] \arrow[d, "p_*"', "\simeq"] & \pi_1(\tilde{X}, \tilde{x}) = 1 \arrow[d, "p_*"] \\
\pi_1(U, x) \arrow[r, "\iota_*"'] & \pi_1(X, x)
\end{tikzcd}$$

$\Rightarrow$ homomorphism ι_* must be trivial !

Ex: An example of a space which is not SSC: "Hawaiian earring"

$X = \bigcup_{n \geq 1}$ Circle of radius $1/n$ in $\mathbb{R}^2$ with center $(0, -\frac{1}{n})$ $\subset \mathbb{R}^2$



- "locally simply-connected" is a stronger property than "semi-locally simply-connected".

E.g. - Warsaw circle is not LPC (⇒ not LSC) but has $\pi_1 = 0 \Rightarrow$ SSC

- Cone(Hawaiian Earring) has $\pi_1 = 0$ (⇒ SSC) but not every point has a simply-con subnbhd of any prescribed nbhd.

def A top. space X is "reasonable" if it is locally path-connected and semilocally simply connected (LPC+SSC)

Theorem

A <u>reasonable</u> path connected space X has a univ. covering $\begin{array}{c}\tilde{X}\\ \downarrow p\\ X\end{array}$.

(Munkres, Thm 82.1 ; Hatcher pp. 64-65)

- Construction of the univ. covering for a path-connected reasonable X:

fix $x_0 \in X$. $\tilde{X} := \{\text{paths in } X \text{ starting at } x_0\} / \text{homotopy of paths}$

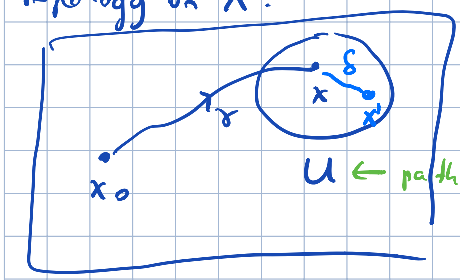
$\downarrow p: \gamma \mapsto \gamma(1)$

X

motivation: $\tilde{X}$ simply conn. $\Rightarrow \exists$ unique homotopy class of paths from $\tilde{x}_0$ to any $\tilde{x}$

- lifting of a path (up to homotopy) in X starting at x_0

topology on $\tilde{X}$:



$U \leftarrow$ path-conn. nbhd s.t. $\pi_1(U) \xrightarrow{i_*} \pi_1(X)$ is trivial

$U_{[\gamma]} := \{[\gamma * \delta] \mid \delta\text{-path in } U \text{ starting at } x\}$

$p: U_{[\gamma]} \to U$

- surjective - since U path-conn
- injective - since two paths x to x' in U are homotopic in X by

- use sets $U_{[\gamma]}$ as a basis for topology on $\tilde{X}$. [Hatcher, p. 64]

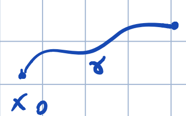
$\langle$ {all path-conn U's with $\pi_1(U) \xrightarrow[i_*]{} \pi_1(X)$ triv.} form a basis for topology on X $\rangle$

- U is evenly covered by $\coprod_{[\gamma]} U_{[\gamma]}$ $\qquad$ - $\tilde{X}$ is path-conn and $\pi_1(\tilde{X}) = 1$

$[\gamma] \leftarrow$ homotopy classes of paths $x_0 \rightsquigarrow x$

- $\tilde{X}$ path-conn?: $[\gamma] \in \tilde{X} \longrightarrow$ path $\Gamma_t = [\gamma(ts) : \overset{s}{I} \to X]$ $\qquad (*)$

$\Gamma_1 = [\gamma]$, $\Gamma_0 = C_{x_0}$



so: Γ is a path from C_{x_0} to $[\gamma]$ ✓

arbitrary pt of $\tilde{X}$

- $\pi_1(\tilde{X}) = 1$? $\iff p_* \pi_1(\tilde{X}) \overset{!}{=} 1 \subset \pi_1(X)$

Let γ based loop in X, x_0. lifting to a loop in $(\tilde{X}, C_{x_0})$

path $t \mapsto [\Gamma_t]$ is the lift of γ to $\tilde{X}$ starting at C_{x_0}.

it is a loop $\overset{(*)}{\Longrightarrow}$ $[\Gamma_1] = [\Gamma_0]$ which proves ✓.

$ \; \overset{\shortparallel}{[\gamma]} \qquad \overset{\shortparallel}{[C_{x_0}]}$

Classification theorem (for covering spaces)

(A) Let X be path-connected, reasonable. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-conn. coverings $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups $H \subset \pi_1(X, x_0)$, obtained as $H = p_* \pi_1(\tilde{X}, \tilde{x}_0)$.

(B) if basepoints are ignored, this correspondence gives a bijection

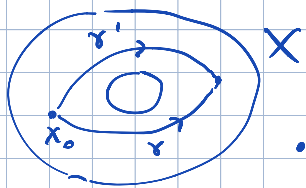
$p: \tilde{X} \to X /_{\text{iso}} \quad \xleftrightarrow{1-1} \quad$ conjugacy classes of subgroups in $\pi_1(X, x_0)$

Proof (A) surjectivity of $\{\text{path-conn, based coverings}\}/_{\text{iso}} \longrightarrow$ subgroups of G. $(*)$

For $H \subset G$ subgrp, construct covering $\begin{matrix} X_H \\ \downarrow p \\ X \end{matrix}$ where $X_H = \widetilde{X} /_{[\gamma] \sim [\gamma']}$ if $\gamma(1) = \gamma'(1)$ and $[\gamma \bar{\gamma}'] \in H$

univ. covering

– it is an equiv. rel., since H is a subgroup.



- $[\gamma] \sim [\gamma'] \iff [\gamma * \delta] \sim [\gamma' * \delta]$

  thus, if two points in basic nbhds $U_{[\gamma]}, U_{[\gamma']}$ are identified in X_H, then the whole neighborhoods are identified.

$\Rightarrow p : X_H \longrightarrow X$ is a covering space
$[\gamma] \longmapsto \gamma(1)$

$$\begin{tikzcd} \widetilde{X} \arrow[r, "q"] \arrow[d, "P_{univ}"'] & X_H \arrow[dl, "p"] \\ X & \end{tikzcd}$$

- choose the basepoint $\tilde{x}_0 = [C_{x_0}] \in X_H$.

$[\gamma] \in p_* \pi_1(X_H, \tilde{x}_0)$ iff (lift to X_H of γ) is a loop

lift of γ to $\widetilde{X}$ starts at C_{x_0}, ends at $[\gamma]$

$\Rightarrow$ lift of γ to X_H is a loop iff $[\gamma] \sim [C_{x_0}]$ or equivalently if $[\gamma] \in H$

$\Rightarrow p_* \pi_1(X_H, \tilde{x}_0) = H \subset G$ ✓

---

injectivity of $(*)$

(– Uniqueness of a covering corresp. to H up to iso)

– if we have iso coverings

$$\begin{tikzcd} \widetilde{X}_1, \tilde{x}_1 \arrow[r, "f"] \arrow[dr, "p_1"'] & \widetilde{X}_2, \tilde{x}_2 \arrow[d, "p_2"] \\ & X, x_0 \end{tikzcd}$$

then $\mathrm{im}(p_1)_* = \mathrm{im}((p_2)_* \circ \underbrace{f_*}_{\text{iso of groups}}) = \mathrm{im}(p_2)_*$

conversely:
if $\mathrm{im}(p_1)_* = \mathrm{im}(p_2)_*$ <but we don't know about f>

$$\begin{tikzcd} \widetilde{X}_1, \tilde{x}_1 \arrow[rr, dashed, bend left, "\widetilde{p}_1"] \arrow[dr, "p_1"'] & & \widetilde{X}_2, \tilde{x}_2 \arrow[ll, dashed, bend left, "\widetilde{p}_2"] \arrow[dl, "p_2"] \\ & X, x_0 & \end{tikzcd}$$

then by lifting criterion, we can lift p_1 to $\widetilde{p}_1$ s.t. $p_2 \circ \widetilde{p}_1 = p_1$. Likewise, we lift p_2 to $\widetilde{p}_2$ s.t. $p_1 \circ \widetilde{p}_2 = p_2$.

$\widetilde{p}_2 \circ \widetilde{p}_1 = id_{\widetilde{X}_1}$ by uniqueness of lifting

and $\widetilde{p}_1 \circ \widetilde{p}_2 = id_{\widetilde{X}_2}$ — " —

$$\begin{tikzcd} \widetilde{X}_1 \arrow[r, shift left, "id"] \arrow[r, shift right, "\widetilde{p}_2 \circ \widetilde{p}_1"'] \arrow[dr, "p_1"'] & \widetilde{X}_1 \arrow[d, "p_1"] \\ & X \end{tikzcd}$$

(B) we already know that changing from a basepoint $\tilde{x}_0$ to a basepoint $\tilde{x}_1 \in p^{-1}(x_0)$ conjugates H.

conversely: if we have $\begin{matrix} \widetilde{X}, \tilde{x}_0 \\ p \downarrow \\ X, x_0 \end{matrix}$ with $p_* \pi_1(\widetilde{X}, \tilde{x}_0) = H_0$ and we want to get $H_1 = g^{-1} H_0 g$ for some $\underset{[\gamma]}{g} \in \pi_1(X, x_0)$, set $\tilde{x}_1 := \underset{\text{lift starting at } \tilde{x}_0}{\tilde{\gamma}}(1)$. Then $p_* \pi_1(\widetilde{X}, \tilde{x}_1) = H_1$.

$\square$

- Group actions

def Let G be a topological group.

(i) A G-action on a space X is a cont. map $G \times X \xrightarrow{\mu} X$, $g, x \longmapsto gx$

satisfying $g(h\,x) = (gh)x$ for $g, h \in G$, $x \in X$.

- If G is a discrete group, continuity of μ is equiv. to continuity of

$$\mu(g,-) : X \longrightarrow X, \quad x \longmapsto gx \quad , \forall g \in G$$

(ii) The action is free if $\forall x \in X$, $gx = x \Leftrightarrow \underset{\text{unit in } G}{g = 1}$

(iii) Action is transitive if $\forall\, x, y \in X$ $\exists g \in G$ s.t. $g \cdot x = y$

(iv) For $x \in X$, the subset $Gx := \{gx \mid g \in G\} \subset X$ is the "orbit through x."

$G\backslash X = \{\text{orbits}\}$ is the orbit space of the G-action on X.

Topology on $G\backslash X$ - the quotient topology determined by $p : X \longrightarrow G\backslash X$, $x \longmapsto Gx$

Examples of group actions

(1) $\mathbb{Z}$ acts on $\mathbb{R}$ via $(n, t) \longmapsto n + t$. Orbit space $\mathbb{Z}\backslash\mathbb{R} \approx S^1$, $[t] \mapsto e^{2\pi i t}$

Thus, the proj. map $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \xrightarrow{\approx} S^1$, $t \longmapsto e^{2\pi i t}$ is our stand. covering of S^1.

(2) $\mathbb{Z}_2 = \{\pm 1\}$ acts on S^n via $\{\pm 1\} \times S^n \longrightarrow S^n$, $(\pm 1, x) \longmapsto \pm x$. Orbit space: $\{\pm 1\}\backslash S^n = \mathbb{R}P^2$.

- in these examples, $p : X \to G\backslash X$ are the univ. coverings for the quotient S^1 or $\mathbb{R}P^2$.

G is π_1 of the quotient.

Lemma: Let $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be the univ. covering of a path-conn, reasonable space X.

Then $\underset{G=}{\pi_1(X, x_0)}$ acts freely on $\tilde{X}$ so that G-orbits are the fibers of p.

def Let G be a group and X a space. A G-covering of X is a covering $p: \tilde{X} \to X$ together with an action of G on $\tilde{X}$ s.t.

(1) $p(g\tilde{x}) = p(\tilde{x}) \quad \forall g \in G, \tilde{x} \in \tilde{X}$ — in particular, the action restricts to an action on each fiber $p^{-1}(x) \subset \tilde{X}$

(2) The action on each fiber is <u>transitive</u>: $\forall \tilde{x}, \tilde{x}' \in p^{-1}(x)$, $\exists g \in G$ s.t. $\tilde{x}' = g\tilde{x}$.

(3) The action is <u>free</u>: if $g\tilde{x} = \tilde{x}$ for some $\tilde{x}$, then $g = 1$

<u>Ex</u>: $S^1 \xrightarrow{p} S^1$, $z \longmapsto z^n$, $n \geq 1$ is a G-covering with $G = \mathbb{Z}_n = \{e^{2\pi i k/n}, k = 0 \dots n-1\}$ — n-th roots of unity

group action:

$z \mapsto g \cdot z = e^{2\pi i k/n} z$, $g = e^{2\pi i k/n}$ — satisfies (1), (2), (3) above

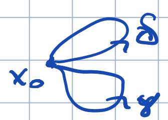
<u>Theorem</u> Let (X, x_0) be a reasonable path-connected space. Then there is a bijection

$$\{\text{based } G\text{-coverings } p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)\}/_{\text{iso}} \longleftrightarrow \left\{\begin{array}{c}\text{group homomorphisms} \\ \varphi: \pi_1(X, x_0) \to G\end{array}\right\}$$

It is given by $p \overset{\Xi}{\longmapsto} \big(\varphi: \underset{\pi_1(X,x_0)}{[\gamma]} \longmapsto g \in G$ s.t. $\tilde{\gamma}(1) = g \cdot \tilde{x}_0\big)$

$\tilde{\gamma}$: lift of γ starting at $\tilde{x}_0$

• proof that φ is a homomorphism



two loops in X

$\tilde{\gamma}$ – lifted path from $\tilde{x}_0$ to $g\tilde{x}_0$, $g = \varphi([\gamma])$

$\tilde{\delta}$ – lifted path from $\tilde{x}_0$ to $h\tilde{x}_0$, $h = \varphi([\delta])$

$\varphi([\gamma][\delta]) = \varphi([\gamma * \delta]) = gh$ ✓

$\widetilde{\gamma * \delta} = \tilde{\gamma} * g(\tilde{\delta})$

– from $\tilde{x}_0$ to $gh\tilde{x}_0$

• <u>construction of Ξ^{-1}</u>:

$$X_G = G \times \tilde{X} \Big/ (g, [\gamma]) \sim (g\varphi(h)^{-1}, h \cdot [\gamma]) \quad \text{for any } h \in \pi_1(X, x_0)$$

$[\gamma]$: path from x_0 in X

$$\begin{array}{ccc} (g, [\gamma]) & & X_G \\ \big\downarrow & & \big\downarrow p \\ \gamma(1) & & X \end{array}$$

• for $p: \tilde{X} \to X$ a covering, isomorphisms
$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$
are called deck transformations (or covering transformations)

- they form a group $\mathrm{Deck}(\tilde{X})$ under composition.

by unique lifting property, a deck transf is fully determined by where it sends a single point! (assuming $\tilde{X}$ is PC)

• only the identity transf. can fix a point in $\tilde{X}$.

• A covering $p: \tilde{X} \to X$ is "normal" (or "regular") if $\mathrm{Deck}(\tilde{X})$ acts transitively in the fibers of p.

• A PC G-covering of X = a normal covering with Deck = G.

Proposition (Hatcher 1.39, p.71) Let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a PC covering of a PC, regular X; let $H = \mathrm{im}\, p_*$. Then

(a) This covering is normal iff H is a normal subgroup of $\pi_1(X, x_0)$.

(b) $\mathrm{Deck}(\tilde{X}) \cong \pi_1(X, x_0)/H$ if the covering is normal. (for $\tilde{X}$ non-normal, $\mathrm{Deck}(\tilde{X}) = N(H)/H$)

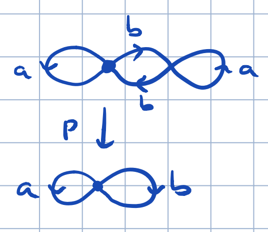
(i.e. for $\tilde{X}$ normal + PC, one has a short exact sequence of groups

$$1 \to \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{p_*} \pi_1(X, x_0) \to \mathrm{Deck}(\tilde{X}) \to 1$$
)

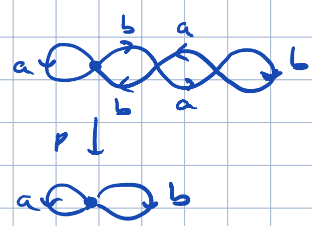
if $\tilde{X}$ non-PC, this map is not surjective!

$$[\gamma] \longmapsto \begin{tikzcd} \tilde{X}, \tilde{x}_0 \arrow[r, "\varphi"] \arrow[dr, "p"'] & \tilde{X}, \tilde{x}_1 = \tilde{\gamma}(1) \arrow[d, "p"] \\ & X, x_0 \end{tikzcd}$$

Ex:



- normal cover ($\mathrm{Deck} = \mathbb{Z}_2$, acting transitively in p-fibers)



- non-normal cover ($\mathrm{Deck} = 1$, acting non-transitively)