• Group actions

def Let G be a topological group.

(i) A G-action on a space X is a cont. map
$$\begin{tikzcd} G\times X \arrow[r, "\mu"] & X \\ g, x \arrow[r, mapsto] & gx \end{tikzcd}$$
satisfying $g(hx) = (gh)x$ for $g, h \in G$, $x \in X$.

- If G is a discrete group, continuity of μ is equiv. to continuity of
$$\mu(g,-) : X \longrightarrow X, \quad x \longmapsto gx \quad , \forall g \in G$$

(ii) The action is free if $\forall x \in X$, $gx = x \iff g = \underset{\uparrow \text{ unit in } G}{1}$

(iii) Action is transitive if $\forall\, x, y \in X$ $\exists g \in G$ s.t. $g \cdot x = y$

(iv) For $x \in X$, the subset $Gx := \{gx \mid g \in G\} \subset X$ is the "orbit through x."

$G\backslash X = \{\text{orbits}\}$ is the orbit space of the G-action on X.

Topology on $G\backslash X$ - the quotient topology determined by $p : X \longrightarrow G\backslash X$, $x \longmapsto Gx$

Examples of group actions

(1) $\mathbb{Z}$ acts on $\mathbb{R}$ via $(n, t) \longmapsto n + t$. Orbit space $\mathbb{Z}\backslash\mathbb{R} \approx S^1$, $[t] \mapsto e^{2\pi i t}$

Thus, the proj. map $\mathbb{R} \longrightarrow \mathbb{Z}\backslash\mathbb{R} \xrightarrow{\approx} S^1$, $t \longmapsto e^{2\pi i t}$ is our stand. covering of S^1.

(2) $\mathbb{Z}_2 = \{\pm 1\}$ acts on S^n via $\{\pm 1\} \times S^n \longrightarrow S^n$, $(\pm 1, x) \longmapsto \pm x$. Orbit space: $\{\pm 1\}\backslash S^n = \mathbb{R}P^2$.

- in these examples, $p : X \to G\backslash X$ are the univ. coverings for the quotient S^1 or $\mathbb{R}P^2$.
  G is π_1 of the quotient.

Lemma: Let $p : (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$ be the univ. covering of a path-conn, reasonable space X. Then $G = \pi_1(X, x_0)$ acts freely on $\widetilde{X}$ so that G-orbits are the fibers of p.

$$\underset{\substack{\uparrow \\ \text{loop}}}{[\gamma]} : \underset{\text{path}}{[\beta]} \longmapsto [\gamma * \beta]$$

def Let G be a group and X a space. A G-covering of X is a covering $p: \tilde{X} \to X$ together with an action of G on $\tilde{X}$ s.t.

(1) $p(g\tilde{x}) = p(\tilde{x})$ $\forall g \in G, \tilde{x} \in \tilde{X}$ - in particular, the action restricts to an action on each fiber $p^{-1}(x) \subset \tilde{X}$

(2) The action on each fiber is **transitive**: $\forall \tilde{x}, \tilde{x}' \in p^{-1}(x)$, $\exists g \in G$ s.t. $\tilde{x}' = g\tilde{x}$.

(3) The action is **free**: if $g\tilde{x} = \tilde{x}$ for some $\tilde{x}$, then $g = 1$

Ex:

$$\begin{tikzcd} S^1 \arrow[r, "p"] & S^1 \\ z \arrow[r, mapsto] & z^n \end{tikzcd}$$

, $n \geq 1$ is a G-covering with $G = \mathbb{Z}_n$
$= \{e^{2\pi i k/n}, k = 0 \dots n-1\}$
$\uparrow$ n-th roots of unity

group action:
$z \mapsto \underset{\overset{\shortparallel}{e^{2\pi i k/n}}}{g} \cdot z = e^{2\pi i k/n} z$ - satisfies (1), (2), (3) above

Theorem $(*)$ Let (X, x_0) be a reasonable path-connected space. Then there is a bijection

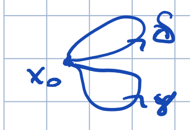
$$\{\text{based } G\text{-coverings } p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)\} / _{\text{iso}} \longleftrightarrow \left\{\begin{matrix} \text{group homomorphisms} \\ \varphi: \pi_1(X, x_0) \to G \end{matrix}\right\}$$

$\nwarrow$ "holonomy homomorphism"

It is given by $p \overset{\longmapsto}{\underset{\Xi}{}} \Big(\varphi: \underset{\underset{\pi_1(X,x_0)}{\cap}}{[\gamma]} \longmapsto g \in G \text{ s.t. } \tilde{\gamma}(1) = g \cdot \tilde{x}_0\Big)$

$\nearrow$ lift of γ starting at $\tilde{x}_0$

- proof that φ is a homomorphism



two loops in X

$\tilde{\gamma}$ - lifted path from $\tilde{x}_0$ to $g\tilde{x}_0$ ($\swarrow$ $\varphi([\gamma])$)

$\tilde{\delta}$ - lifted path from $\tilde{x}_0$ to $h\tilde{x}_0$ ($\nwarrow$ $\varphi([\delta])$)

$\varphi([\gamma][\delta]) = \varphi([\gamma * \delta]) = gh$ ✓

$\widetilde{\gamma * \delta} = \tilde{\gamma} * g(\tilde{\delta})$
- from $\tilde{x}_0$ to $gh\tilde{x}_0$

$t \mapsto (1, [A_t])$, $1 \mapsto (1, [\alpha]) \sim (\varphi([\alpha]), [c_{x_0}])$

- construction of Ξ^{-1}:

$$\begin{tikzcd} (g, [\gamma]) \arrow[d, mapsto] & X_G \arrow[d, dashed, "p"] \\ \gamma(1) & X \end{tikzcd}$$

$X_G = G \times \tilde{X} \Big/ (g, [\gamma]) \sim (g\varphi(h)^{-1}, h \cdot [\gamma])$

$\uparrow$ path from x_0 in X for any $h \in \pi_1(X, x_0)$

loop $\alpha: I \to X$

Remark: 1) if $\begin{array}{c}\tilde{X},\tilde{x}_0\\ p\downarrow \\ X, x_0\end{array}$ - G-covering, $\underset{x_0}{\overset{\psi}{Y}} \overset{i}{\hookrightarrow} X$, then one has a restricted covering over Y $\begin{array}{c}\tilde{X}|_Y,\tilde{x}_0\\ \downarrow p|_Y \\ Y, x_0\end{array}$.

The bijection of THM(*) is functorial wrt restrictions:

$$\begin{tikzcd}\pi_1(Y,x_0) \arrow[r, "i_*"] \arrow[rr, bend right, "\varphi_{p|_Y}"'] & \pi_1(X,x_0) \arrow[r, "\varphi_p"'] & G\end{tikzcd}$$

2) in THM(*), φ is <u>surjective</u> iff the covering $\tilde{X}$ is <u>path-connected</u>.

- for $p: \tilde{X} \to X$ a covering, isomorphisms

$$\begin{tikzcd}\tilde{X} \arrow[rr, "\textcircled{f}"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

are called <u>deck transformations</u> (or <u>covering transformations</u>)

- they form a group $\mathrm{Deck}(\tilde{X})$ under composition.

by unique lifting property, a deck transf is fully determined by where it sends a single point! (assuming $\tilde{X}$ is PC)

- only the identity transf. can fix a point in $\tilde{X}$.

- A covering $p: \tilde{X} \to X$ is "normal" (or "regular") if $\mathrm{Deck}(\tilde{X})$ acts <u>transitively</u> in the fibers of p.

- A $\overset{PC}{\vee}$ G-covering of X = a normal covering with Deck = G.

<u>Proposition</u> (Hatcher 1.39, p.71) Let $p: (\tilde{X},\tilde{x}_0) \to (X,x_0)$ be a PC covering of a PC, regular X; let $H = \mathrm{im}\, p_*$. Then

(a) This covering is normal iff H is a <u>normal subgroup</u> of $\pi_1(X,x_0)$.

(b) $\mathrm{Deck}(\tilde{X}) \simeq \pi_1(X,x_0)/H$ if the covering is normal. (for $\tilde{X}$ non-normal, $\mathrm{Deck}(\tilde{X}) \simeq N(H)/H$)

(i.e. for $\tilde{X}$ normal$\overset{+PC}{,}$ one has a short exact sequence of groups

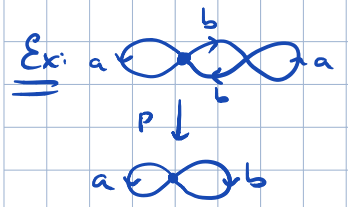
$$1 \to \pi_1(\tilde{X},\tilde{x}_0) \xrightarrow[p_*]{} \pi_1(X,x_0) \to \mathrm{Deck}(\tilde{X}) \to 1$$

if $\tilde{X}$ non-PC, this map is not surjective!

$$[\gamma] \longmapsto \begin{tikzcd}\tilde{X},\tilde{x}_0 \arrow[r, "\boxed{\tilde{\gamma}}"] \arrow[dr, "p"'] & \tilde{X}, \tilde{x}_1 = \tilde{\gamma}(1) \arrow[d, "p"] \\ & X, x_0\end{tikzcd}$$

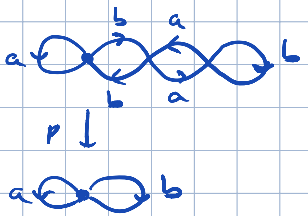
)

Ex:



- normal cover

(Deck = $\mathbb{Z}_2$, acting transitively in p-fibers)



- non-normal cover

(Deck = 1, acting non-transitively)

Seifert-van Kampen (for reasonable spaces)

Let X be a top. space,

$X = \underset{\substack{\uparrow \\ \text{open}}}{X_1} \cup \underset{\uparrow}{X_2}$, $X_1 \cap X_2 = \underset{\substack{\rotatebox{90}{$\in$} \\ x_0}}{Y}$

assume: X, X_1, X_2, Y are path-connected and reasonable.

Then

$$\begin{tikzcd}
\pi_1(Y, x_0) \arrow[r, "(j_1)_*"] \arrow[d, "(j_2)_*"'] & \pi_1(X_1, x_0) \arrow[d, "(k_1)_*"] \\
\pi_1(X_2, x_0) \arrow[r, "(k_2)_*"'] & \pi_1(X, x_0)
\end{tikzcd}$$

is a pushout diagram.

Here $j_a : Y \hookrightarrow X_a$, $k_a : X_a \hookrightarrow X$ inclusions, $a = 1, 2$.

Sketch of Proof Need to show that, given a group G and maps $f_1, f_2 : \pi_1(X_a, x_0) \to G$ agreeing on $\pi_1(Y, x_0)$, we can construct $\overset{\text{unique}}{\text{a map}}$ $f : \pi_1(X, x_0) \to G$

s.t.

(@)
$$\begin{tikzcd}
\pi_1(Y, x_0) \arrow[r] \arrow[d] & \pi_1(X_1, x_0) \arrow[d] \arrow[ddr, bend left, "f_1"] & \\
\pi_1(X_2, x_0) \arrow[r] \arrow[drr, bend right, "f_2"'] & \pi_1(X, x_0) \arrow[dr, dashed, "f"] & \\
 & & G
\end{tikzcd}$$

commutes.

By Thm (*), f_1, f_2 give G-coverings $\begin{tikzcd} \tilde{X}_1, \tilde{x}_1 \arrow[d, "p_1"] \\ X_1, x_0 \end{tikzcd}$, $\begin{tikzcd} \tilde{X}_2, \tilde{x}_2 \arrow[d, "p_2"] \\ X_2, x_0 \end{tikzcd}$ which restrict to isomorphic

G-coverings
$$\begin{tikzcd}
p_1^{-1}(Y), \tilde{x}_1 \arrow[r, "\psi \leftarrow \text{iso}"] \arrow[d, "p_1|_Y"'] & p_2^{-1}(Y), \tilde{x}_2 \arrow[d, "p_2|_Y"] \\
Y, x_0 & Y, x_0
\end{tikzcd}$$

Using ψ, we can glue

$$\begin{tikzcd}
\tilde{X}_1 \underset{\psi}{\cup} \tilde{X}_2 =: \tilde{X}, \tilde{x}_0 \arrow[d, "p"] \\
X, x_0
\end{tikzcd}$$

- this is a G-covering

=> by THM (*) it induces a homomorphism $f : \pi_1(X, x_0) \to G$ making (@) commute.

Uniqueness follows from uniqueness of the construction in each step. ☐