• for $p:\tilde{X}\to X$ a covering, isomorphisms
$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$
are called <u>deck transformations</u> (or <u>covering transformations</u>)

- they form a group $\mathrm{Deck}(\tilde{X})$ under composition.

by unique lifting property, a deck transf. is fully determined by where it sends a single point! (assuming $\tilde{X}$ is PC)

• only the identity transf. can fix a point in $\tilde{X}$.

• A covering $p:\tilde{X}\to X$ is "normal" (or "regular") if $\mathrm{Deck}(\tilde{X})$ acts <u>transitively</u> in the fibers of p.

• A $\overset{PC}{\vee}$ G-covering of X = a normal covering with Deck = G.

<u>Proposition</u> (Hatcher 1.39, p.71) Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a PC covering of a PC, regular X; let $H=\mathrm{im}\,p_*$. Then

(a) This covering is normal iff H is a <u>normal subgroup</u> of $\pi_1(X,x_0)$.

(b) $\mathrm{Deck}(\tilde{X})\simeq \pi_1(X,x_0)/H$ if the covering is normal. (for $\tilde{X}$ non-normal, $\mathrm{Deck}(\tilde{X})\simeq N(H)/H$)

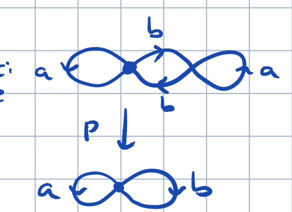
(i.e. for $\tilde{X}$ normal, +PC one has a short exact sequence of groups

$$1\longrightarrow \pi_1(\tilde{X},\tilde{x}_0)\xrightarrow{p_*}\pi_1(X,x_0)\longrightarrow \mathrm{Deck}(\tilde{X})\longrightarrow 1$$

if $\tilde{X}$ non-PC, this map is not surjective!

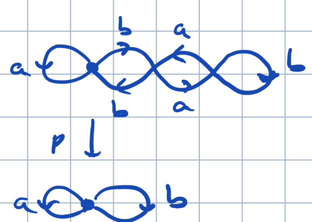
$$[\gamma]\longmapsto \begin{tikzcd} \tilde{X},\tilde{x}_0 \arrow[r, "\tilde{\gamma}"] \arrow[dr, "p"'] & \tilde{X},\ \tilde{x}_1=\tilde{\gamma}(1) \arrow[d, "p"] \\ & X,x_0 \end{tikzcd}$$
)

<u>Ex:</u>



- normal cover

($\mathrm{Deck}=\mathbb{Z}_2$, acting transitively in p-fibers)



- non-normal cover

($\mathrm{Deck}=1$, acting non-transitively)

Seifert-van Kampen (for reasonable spaces)

Let X be a top. space,

$X = X_1 \cup X_2$ (open), $X_1 \cap X_2 = Y \ni x_0$

assume: X, X_1, X_2, Y are path-connected and reasonable.

Then

$$\begin{tikzcd} \pi_1(Y, x_0) \arrow[r, "(j_1)_*"] \arrow[d, "(j_2)_*"'] & \pi_1(X_1, x_0) \arrow[d, "(k_1)_*"] \\ \pi_1(X_2, x_0) \arrow[r, "(k_2)_*"'] & \pi_1(X, x_0) \end{tikzcd}$$

is a pushout diagram.

Here $j_a : Y \hookrightarrow X_a$, $k_a : X_a \hookrightarrow X$ inclusions, $a = 1, 2$.

Sketch of Proof: Need to show that, given a group G and maps $f_1, f_2 : \pi_1(X_a, x_0) \to G$ agreeing on $\pi_1(Y, x_0)$, we can construct a unique map $f : \pi_1(X, x_0) \to G$

s.t.

(@)

$$\begin{tikzcd} \pi_1(Y, x_0) \arrow[r] \arrow[d] & \pi_1(X_1, x_0) \arrow[d] \arrow[ddr, bend left, "f_1"] & \\ \pi_1(X_2, x_0) \arrow[r] \arrow[drr, bend right, "f_2"'] & \pi_1(X, x_0) \arrow[dr, dashed, "f"] & \\ & & G \end{tikzcd}$$

commutes.

By Thm (*), f_1, f_2 give G-coverings $\begin{matrix} \tilde{X}_1, \tilde{x}_1 \\ p_1 \downarrow \\ X_1, x_0 \end{matrix}$, $\begin{matrix} \tilde{X}_2, \tilde{x}_2 \\ p_2 \downarrow \\ X_2, x_0 \end{matrix}$ which restrict to isomorphic G-coverings

$$\begin{tikzcd} p_1^{-1}(Y), \tilde{x}_1 \arrow[r, "\psi \leftarrow \text{iso}"] \arrow[d, "p_1|_Y"'] & p_2^{-1}(Y), \tilde{x}_2 \arrow[d, "p_2|_Y"] \\ Y, x_0 & Y, x_0 \end{tikzcd}$$

Using ψ, we can glue

$$\begin{tikzcd} \tilde{X}_1 \cup_{\psi} \tilde{X}_2 =: \tilde{X}, \tilde{x}_0 \arrow[d, "p"] \\ X, x_0 \end{tikzcd}$$

- this is a G-covering

=> by THM (*) it induces a homomorphism $f : \pi_1(X, x_0) \to G$ making (@) commute.

Uniqueness follows from uniqueness of the construction in each step. $\square$

Smooth manifolds

motivation: M - space of configurations of a physical system (e.g. Anglepoise lamp)
- particle moving on S^2: want to solve an ODE there (diff. "equation of motion")
⟨- spacetime :- general relativity⟩

+ want to be able to do calculus on M - differentiate & integrate functions.
- what are smooth functions? • which objects can be integrated over M?
- what kind of object is df?

def Let M be a topological n-manifold. A (coordinate) chart for M is an open subset $U \subset M$ and a homeomorphism $\varphi: U \to \varphi(U) \underset{\text{open}}{\subset} \mathbb{R}^n$

An atlas on M is a collection of coordinate charts $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ s.t.

- M is covered by $\{U_\alpha\}_{\alpha\in I}$
- $\forall \alpha,\beta \in I$, the map $\varphi_\beta \varphi_\alpha^{-1}: \underbrace{\varphi_\alpha(U_\alpha \cap U_\beta)}_{\text{open}} \to \underbrace{\varphi_\beta(U_\alpha\cap U_\beta)}_{\text{open}}$ is C^∞

⟨recall that $F: \underset{\mathbb{R}^n}{V} \to \mathbb{R}$, $(x_1,\dots,x_n) \mapsto F(x_1,\dots,x_n)$ is "smooth" or C^∞ if it has partial derivatives of all orders

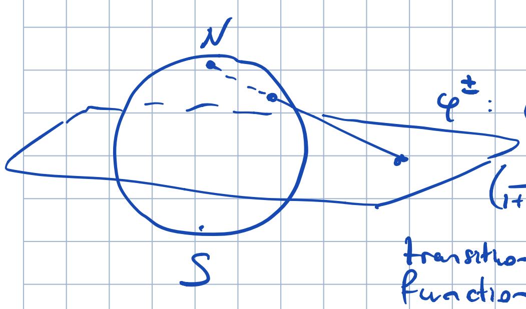
likewise, $F: V \to \mathbb{R}^m$ is C^∞ if each component has all derivatives ⟩

Examples (charts & atlases)

- $\underset{\text{open}}{U \subset \mathbb{R}^n}$, $\varphi = id$ - tautologically
- $M = S^n$ (unit sphere in $\mathbb{R}^{n+1}$) covered by $U_i^+ = \{(x_0,\dots,x_n)\in S^n \mid x_i > 0\}$, $U_i^- = \{(x_0,\dots,x_n)\in S^n \mid x_i < 0\}$, $i=0,\dots,n$

$\varphi_i^\pm: U_i^\pm \to \mathbb{R}^n$, $(x_0,\dots,x_n) \mapsto (x_0,\dots,\hat{x}_i,\dots,x_n)$, $\operatorname{im}\varphi_i^\pm =$ open unit ball $B_1(0)\subset\mathbb{R}^n$

- $M = S^n$ with $\varphi_i^\pm: U^\pm = S^n \setminus \underbrace{\{(0,\dots,0,\pm1)\}}_{\text{"North/South poles"}} \to \mathbb{R}^n$ stereographic projections



$\varphi^\pm: (x_0,\dots,x_n) \longmapsto \frac{1}{1\mp x_n}(x_0,\dots,x_{n-1})$

$\left(\frac{2}{1+\|v\|^2}\vec{v}, \pm\frac{\|v\|^2-1}{\|v\|^2+1}\right) \longleftarrow (v_1,\dots,v_n)$

transition function $\varphi_-\varphi_+^{-1}: (v_1,\dots,v_n) \mapsto \frac{1}{\|v\|^2}(v_1 \dots v_n)$

- $M = \mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} \big/ {\substack{x \sim \lambda x \\ \lambda \in \mathbb{R} \setminus \{0\}}}$. $U_i := \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_i \neq 0\}$

$\varphi_i : U_i \to \mathbb{R}^n$

$[x_0 \cdots x_n] \mapsto \left(\frac{x_0}{x_i}, \dots \widehat{\frac{x_i}{x_i}} \dots \frac{x_n}{x_i}\right)$ - homeomorphism,

$\underset{\uparrow \text{ i-th place}}{[v_1, \dots, 1, \dots v_n]} \xleftarrow{(\varphi_i)^{-1}} (v_1, \dots, v_n)$ - inverse

So, (U_i, φ_i) - atlas for $\mathbb{RP}^n$

$\varphi_i(U_i \cap U_j) = (v_0, \dots, \underset{\uparrow}{1}, \dots v_n)$ with $v_j \neq 0$.

$\varphi_i \varphi_j^{-1} : \{x \in \mathbb{R}^{n+1} \mid x_j = 1, x_i \neq 0\} \to \{x \in \mathbb{R}^{n+1} \mid x_i = 1, x_j \neq 0\}$

$v \longmapsto \frac{1}{x_i} v$

The definition of a manifold

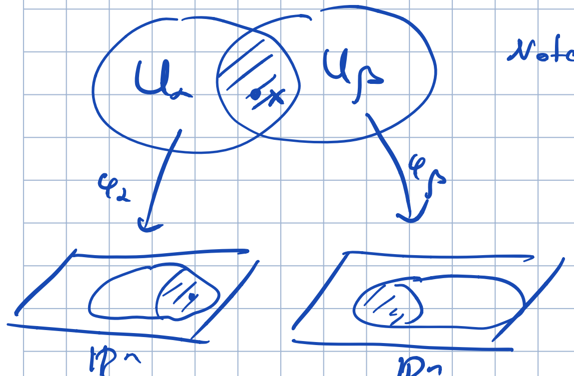
- Two atlases $(U_\alpha, \varphi_\alpha)$, (V_i, ψ_i) on M are compatible if their union is an atlas.
  <i.e. extra maps $\psi_i \varphi_\alpha^{-1}$ must be smooth >

- A smooth structure on M is an equivalence class of atlases.

def An n-dimensional smooth manifold is a topological n-manifold with a smooth structure.

- $f : M \to \mathbb{R}$ is a smooth function if $\forall (U_\alpha, \varphi_\alpha)$ coord. chart of the atlas,

$\mathbb{R}^n \supset \varphi_\alpha(U_\alpha) \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \xrightarrow{f} \mathbb{R}$, with $f \circ \varphi_\alpha^{-1}$ the composite, and $U_\alpha \subset M$, is a smooth function of n-variables



Note: on an overlap $U_\alpha \cap U_\beta$, $f \circ \varphi_\alpha^{-1}$ is smooth at $\varphi_\alpha(x)$ (x in $U_\alpha \cap U_\beta$)

iff $f \circ \varphi_\beta^{-1}$ is smooth at $\varphi_\beta(x)$

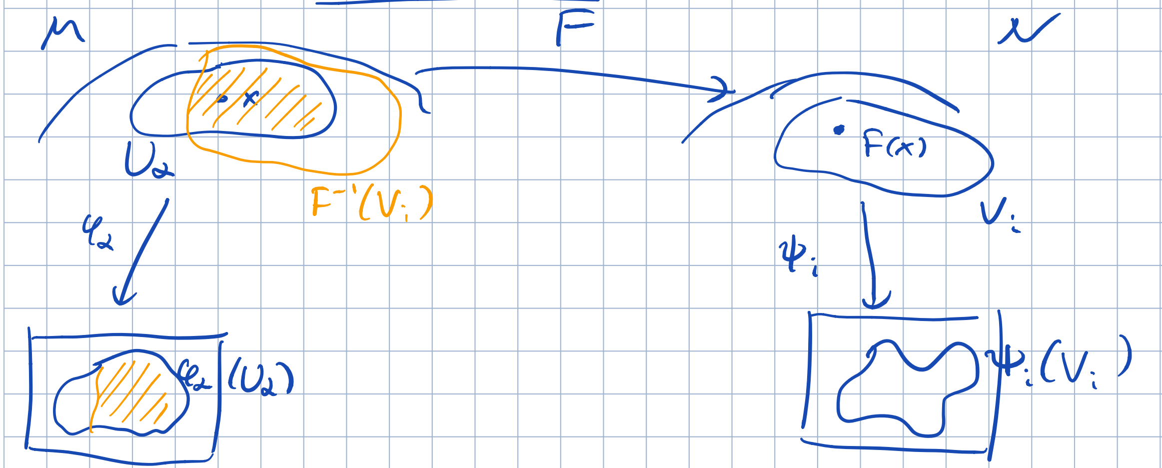
$$f \circ \varphi_\beta^{-1} = (f \circ \varphi_\alpha^{-1}) \circ \underbrace{(\varphi_\alpha \circ \varphi_\beta^{-1})}_{\text{transition map}}$$

• A continuous map $F: M \to N$ of manifolds is a smooth map if

for each $x \in M$ and a chart $(U_\alpha, \varphi_\alpha)$ -chart of M, $U_\alpha \ni x$, and (V_i, ψ_i) -chart of N, $V_i \ni F(x)$,

the function $\psi_i \circ F \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap F^{-1}(V_i)) \longrightarrow \mathbb{R}^n$, where $\varphi_\alpha(U_\alpha \cap F^{-1}(V_i)) \subset \mathbb{R}^n$,

is a C^∞ function. (*)



Rem: it is enough to check (*) for one atlas - it is then automatically true in any compatible atlas (since $\varphi_\alpha \varphi_\beta^{-1}$ are C^∞)

* A diffeomorphism $F: M \to N$ is a smooth map with smooth inverse.