Smooth manifolds

motivation: M - space of configurations of a physical system (e.g. Anglepoise lamp)
- particle moving on S^2: want to solve an ODE there (diff. "equation of motion")
<- spacetime - general relativity>

+ want to be able to do calculus on M - differentiate & integrate functions.

- what are smooth functions?
- which objects can be integrated over M?
- what kind of object is df?

def Let M be a topological n-manifold. A (coordinate) chart for M is an open subset $U \subset M$ and a homeomorphism $\varphi : U \to \varphi(U) \underset{\text{open}}{\subset} \mathbb{R}^n$

An **atlas** on M is a collection of coordinate charts $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ s.t.

- M is covered by $\{U_\alpha\}_{\alpha \in I}$
- $\forall \alpha, \beta \in I$, the map $\varphi_\beta \varphi_\alpha^{-1} : \underbrace{\varphi_\alpha(U_\alpha \cap U_\beta)}_{\text{open}} \to \underbrace{\varphi_\beta(U_\alpha \cap U_\beta)}_{\text{open}}$ is C^∞

<recall that $F : \underset{\mathbb{R}^n}{\overset{}{V}} \to \mathbb{R}$, $(x_1, \dots, x_n) \mapsto F(x_1, \dots, x_n)$ (with $V \subset \mathbb{R}^n$) is "smooth" or C^∞ if it has partial derivatives of all orders

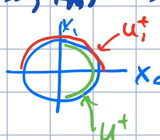
likewise, $F : V \to \mathbb{R}^m$ is C^∞ if each component has all derivatives >

Examples (charts & atlases)

- $\underset{\text{open}}{U \subset \mathbb{R}^n}$, $\varphi = id$ - tautologically
- $M = S^n$ (unit sphere in $\mathbb{R}^{n+1}$) covered by
$$U_i^+ = \{(x_0, \dots, x_n) \in S^n \mid x_i > 0\}, \qquad i = 0, \dots, n$$
$$U_i^- = \{(x_0, \dots, x_n) \in S^n \mid x_i < 0\}$$

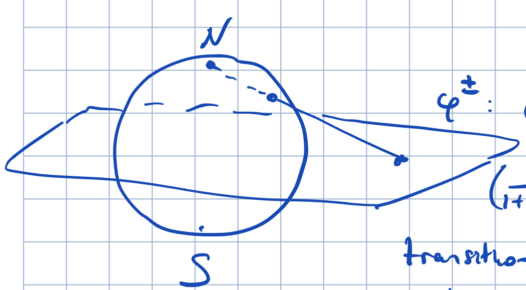
$$\varphi_i^\pm : U_i^\pm \to \mathbb{R}^n, \quad (x_0, \dots, x_n) \mapsto (x_0, \dots, \hat{x}_i, \dots, x_n)$$
, im $\varphi_i^\pm$ = open unit ball $B_1(0) \subset \mathbb{R}^n$

$n=1$:



$\varphi_1 \circ \varphi_0^{-1} : x_0 \mapsto x_1 = \sqrt{1-(x_0)^2}$ ($x_0 \in (0,1)$) - smooth transition map (one of)

- $M = S^n$ with $\varphi_\pm : U^\pm = S^n \setminus \underbrace{\{(0, \dots, 0, \pm 1)\}}_{\text{"North/South poles"}} \to \mathbb{R}^n$ stereographic projections



$$\varphi^\pm : (x_0, \dots, x_n) \longmapsto \frac{1}{1 \mp x_n}(x_0, \dots, x_{n-1})$$
$$\left(\frac{2}{1+\|v\|^2}\vec{v}, \pm\frac{\|v\|^2-1}{\|v\|^2+1}\right) \longleftarrow (v_1, \dots, v_n)$$

transition map $\varphi_- \varphi_+^{-1} : (v_1, \dots, v_n) \mapsto \frac{1}{\|v\|^2}(v_1 \dots v_n)$

- $M = \mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} \Big/ {\scriptstyle x \sim \lambda x, \ \lambda \in \mathbb{R}\setminus\{0\}}$. $\quad U_i := \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_i \neq 0\}$

$\varphi_i : U_i \longrightarrow \mathbb{R}^n$

$[x_0 \cdots x_n] \longmapsto \left(\frac{x_0}{x_i}, \dots \widehat{\frac{x_i}{x_i}} \cdots \frac{x_n}{x_i}\right)$ - homeomorphism,

$\underset{\uparrow \text{ i-th place}}{[v_1, \dots, 1, \dots v_n]} \xleftarrow{(\varphi_i)^{-1}} (v_1, \dots, v_n)$ - inverse

So, (U_i, φ_i) - atlas for $\mathbb{RP}^n$

$\varphi_i(U_i \cap U_j) = (v_0, \dots, 1, \dots v_n)$ with $v_j \neq 0$.

$\varphi_i \varphi_j^{-1} : \{x \in \mathbb{R}^{n+1} \mid x_j = 1, x_i \neq 0\} \longrightarrow \{x \in \mathbb{R}^{n+1} \mid x_i = 1, x_j \neq 0\}$

$v \longmapsto \frac{1}{x_i} v$

The definition of a manifold

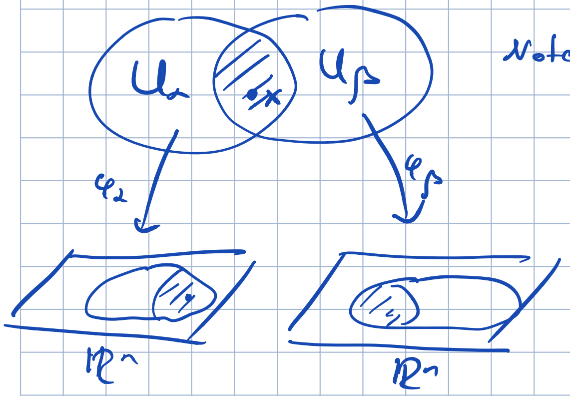
- Two atlases $(U_\alpha, \varphi_\alpha)$, (V_i, ψ_i) on M are compatible if their union is an atlas.
  <i.e. extra maps $\psi_i \varphi_\alpha^{-1}$ must be smooth>

- A smooth structure on M is an equivalence class of atlases.

def An n-dimensional smooth manifold is a topological n-manifold with a smooth structure.

- $f : M \longrightarrow \mathbb{R}$ is a smooth function if $\forall (U_\alpha, \varphi_\alpha)$ coord. chart of the atlas,

$\mathbb{R}^n \supset \varphi_\alpha(U_\alpha) \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \xrightarrow{f} \mathbb{R}$, with $U_\alpha \subset M$ and $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \mathbb{R}$ is a smooth function of n-variables



Note: on an overlap $U_\alpha \cap U_\beta$, $f \circ \varphi_\alpha^{-1}$ is smooth at $\varphi_\alpha(x)$ (in $U_\alpha \cap U_\beta$)

iff $f \circ \varphi_\beta^{-1}$ is smooth at $\varphi_\beta(x)$

$$\parallel$$

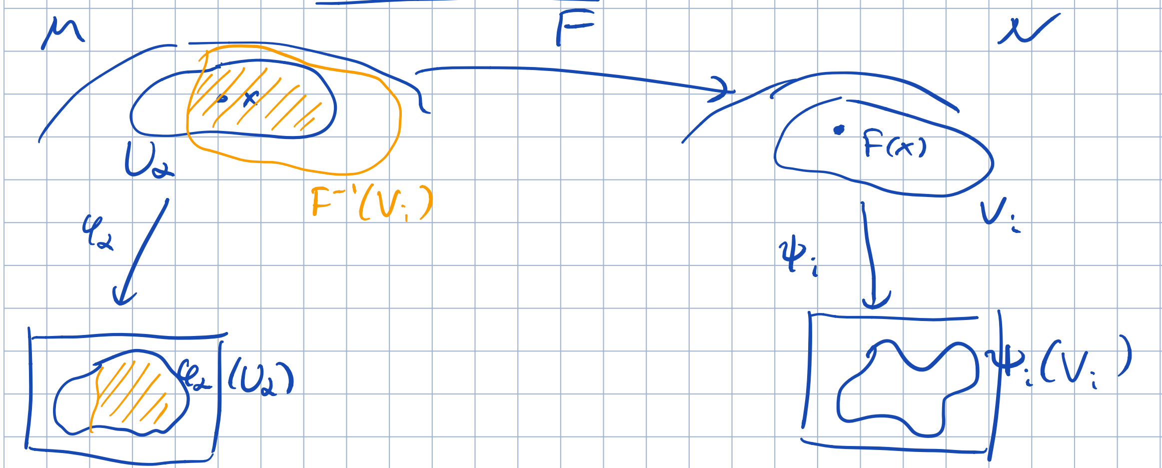
$$(f \circ \varphi_\alpha^{-1}) \circ \underbrace{(\varphi_\alpha \circ \varphi_\beta^{-1})}_{\text{transition map}}$$

• A continuous map $F: M \to N$ of manifolds is a <u>smooth map</u> if

for each $x \in M$ and a chart $(U_\alpha, \varphi_\alpha)$ - chart of M, $U_\alpha \ni x$, and (V_i, ψ_i) - chart of N, $V_i \ni F(x)$,

the function $\psi_i \circ F \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap F^{-1}(V_i)) \longrightarrow \mathbb{R}^m$
$\quad \cap \mathbb{R}^n$

is a <u>C^∞ function</u>. $(*)$



<u>Rem</u>: it is enough to check $(*)$ for one atlas - it is then automatically true in any compatible atlas (since $\varphi_\alpha \varphi_\beta^{-1}$ are C^∞)

* A <u>diffeomorphism</u> $F: M \to N$ is a smooth map with smooth inverse.

chain rule reminder:

$a \in U \xrightarrow{F} V \ni F(a) \xrightarrow{G} W$, $\quad U \subset \mathbb{R}^m,\ V \subset \mathbb{R}^n,\ W \subset \mathbb{R}^k$

$\mathbb{R}^m \xrightarrow{DF_a} \mathbb{R}^n \xrightarrow{DG_{F(a)}} \mathbb{R}^k$

chain rule: $D(G \circ F)_a = DG_{F(a)} \circ DF_a$

or: $\dfrac{\partial z_\alpha}{\partial x_i} = \sum \dfrac{\partial z_\alpha}{\partial y_a} \dfrac{\partial y_a}{\partial x_i}$

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<source of examples of manifolds>

<u>Thm</u>

Let $F: U \to \mathbb{R}^m$, $F = (F_1, \dots, F_m)$, $U \subset \mathbb{R}^{n+m}$ open, be a C^∞ function, fix $c \in \mathbb{R}^m$. Assume that $\forall a \in F^{-1}(c)$,

the derivative $DF_a : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is <u>surjective</u>. Then $M = F^{-1}(c) \subset \mathbb{R}^{n+m}$

$= \left(\dfrac{\partial F_i}{\partial x_j}\right)\Big|_{x=a}$ has the structure of a smooth n-manifold.

recall: $F(a+h) = F(a) + DF_a(h) + R(a,h)$ with $R(a,h)/\|h\| \xrightarrow[h\to 0]{} 0$

<u>Proof</u>: DF_a is surjective $\iff$ matrix $\left(\dfrac{\partial F_i}{\partial x_j}\right)\Big|_{x=a}$, $1 \le i \le m$, $1 \le j \le n+m$, has rank $= m$

$\Rightarrow$ by reordering the coordinates $x_1, \dots, x_{n+m}$, we may assume that the <u>square matrix</u>

$\left(\dfrac{\partial F_i}{\partial x_j}\right)_{\substack{1 \le i \le m \\ 1 \le j \le m}}$ is <u>invertible</u>. Define $G: U \to \mathbb{R}^{n+m}$

$(x_1, \dots, x_{n+m}) \mapsto (F_1, \dots, F_m, x_{m+1}, \dots, x_{n+m})$ $\Rightarrow DG_a$ is <u>invertible</u>

$DG_a = \begin{pmatrix} \text{invertible} & * \\ 0 & I \end{pmatrix} \begin{matrix} \}m \\ \}n \end{matrix}$

By inverse function theorem, $\exists\, \underset{\substack{\text{open}\\ \ni a}}{V} \subset \mathbb{R}^{n+m},\ \underset{\substack{\text{open}\\ \ni G(a)}}{W} \subset \mathbb{R}^{n+m}$ s.t. $G: V \to W$ is invertible, with smooth inverse (4)

G maps $V \cap F^{-1}(c)$ to $\underbrace{(\{c\}\times\mathbb{R}^n)}\cap W$

copy of $\mathbb{R}^n$ given by $\{x \in \mathbb{R}^{n+m};\ x_1=c_1,\dots,x_m=c_m\}$

$\Rightarrow \underbrace{p\circ G}_{\varphi}: V \cap F^{-1}(c) \to \mathbb{R}^n$ is a coord. chart on $M = F^{-1}(c)$.

$V_\alpha \xrightarrow{G_\alpha} W_\alpha$

$V_\beta \xrightarrow{G_\beta} W_\beta$

Given two such charts $\varphi_\alpha, \varphi_\beta$, $\quad G_\alpha G_\beta^{-1}$ - C^∞-map between open sets in $\mathbb{R}^{n+m}$

$\Rightarrow \varphi_\alpha \varphi_\beta^{-1} = p\, G_\alpha G_\beta^{-1} \underbrace{i}$ is C^∞.

inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+m} = \mathbb{R}^m \times \mathbb{R}^n$

$\Rightarrow$ we have an atlas.

• $\mathbb{R}^{n+m}$ is Hausdorff, 2^{nd} countable $\Rightarrow M = F^{-1}(c) \subseteq \mathbb{R}^{n+m}$ is, too $\square$.

Ex: ① $S^n = F^{-1}(1)$ where $F: \mathbb{R}^{n+1} \to \mathbb{R}$, $S^n = F^{-1}(1)$

$x_1,\dots,x_{n+1} \mapsto (x_1)^2 + \dots + (x_{n+1})^2$

$DF_a(h) = \sum_{i=1}^{n} 2a_i h_i \Rightarrow DF_a \neq 0$ iff $a \neq 0$. In particular, $\|a\| = 1 \Rightarrow DF_a \neq 0$.

$\Rightarrow$ Thm applies and S^n is a manifold.

② $O(n) = \{A \in \mathrm{Mat}_{n\times n} \mid A^T A = I\}$

$$F: \underset{\substack{\|\mathcal{\wr}\\ \mathbb{R}^{n^2}}}{\mathrm{Mat}_{n\times n}} \to \underset{\substack{\|\mathcal{\wr}\\ \mathbb{R}^{\frac{n^2+n}{2}}}}{\text{Symmetric } n\times n \text{ matrices}}$$

$A \mapsto A^T A$

$O(n) = F^{-1}(I)$.

$$DF_A(H) = A^T H + H^T A$$

set $H = AK$

$$= A^T A K + K^T A^T A$$

$$= K + K^T \quad \text{if } A \in F^{-1}(I)$$

$\Rightarrow DF_A$ is surjective $\forall A \in F^{-1}(I) \Rightarrow O(n) = F^{-1}(I)$

(taking $K = \frac{S}{2}$ for S any sym. matrix)

is a smooth manifold of dimension $n^2 - \frac{n(n+1)}{2} = \boxed{\frac{n(n-1)}{2}}$

<back to>
Smooth functions

$f : M \to \mathbb{R}$ C^∞ – can add and multiply by constants $\Rightarrow$ they form a vector space $C^\infty(M)$

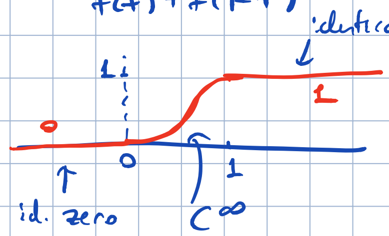
– can multiply themselves $\Rightarrow$ form a commutative ring.

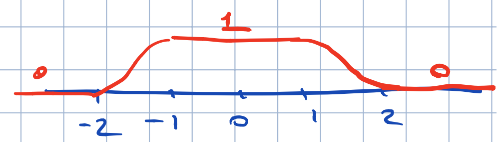
* There are many smooth functions on a smooth mfd M.

· **Bump function**

one-variable:

$$f(t) = \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \le 0 \end{cases} \qquad g(t) = \frac{f(t)}{f(t) + f(1-t)} \qquad h(t) = g(2+t)\,g(2-t) \begin{cases} = 0 & \text{if } |t| \ge 2 \\ = 1 & \text{if } |t| \le 1 \end{cases}$$





• n-dim. version

~~$k(x_1, \dots, x_n) = h(x_1) \cdots h(x_n)$~~ $\quad$ (*) $\quad$ ~~$k\left(\frac{x}{r}\right) = \begin{cases} 1 & \text{in a ball of radius } r \\ 0 & \text{outside a ball of radius } 2\sqrt{n}\, r \end{cases}$~~

BETTER: $k(x_1, \dots, x_n) = h(\|x\|) = h\left(\sqrt{x_1^2 + \dots + x_n^2}\right) \Rightarrow k\left(\frac{x}{r}\right) = \begin{cases} 1 & \text{if } \|x\| \le r \\ 0 & \text{if } \|x\| \ge 2r \end{cases}$

• let (U, φ_U) a coord. chart on M

choose a function k of the type (*) s.t. $\operatorname{supp} k = \overline{\{x : k(x) \neq 0\}}$ lies in $\varphi_U(U)$

and set
$$f : M \to \mathbb{R}$$
$$x \mapsto \begin{cases} k \circ \varphi_U(x), & x \in U \\ 0, & x \in M \setminus U \end{cases}$$

$\underline{\underline{f \in C^\infty(M)}}$: $\Big[$ f is C^∞ in U. $\quad$ $\operatorname{supp} k \subset \mathbb{R}^n$ closed, bounded $\Rightarrow$ compact

$\Rightarrow \operatorname{supp} f \subset M$ compact

$\Rightarrow f = 0$ in $\underbrace{M}_{\text{Hausdorff}} \setminus \underbrace{\operatorname{supp} f}_{\text{cpt} \Rightarrow \text{closed}}$ = open in M $\Rightarrow$ $f = 0$ in a nbhd of any pt $x \in M \setminus \operatorname{supp} f$

$\Rightarrow f$ is smooth in $M \setminus \operatorname{supp} f$

$\Rightarrow \underline{\underline{f \in C^\infty(M)}}$

$\square$

Derivative of a function

$f \in C^\infty(M)$ $\quad$ when does a derivative at a vanish? (e.g. f has a maximum at a)

$g = f \circ \underbrace{\varphi_\alpha^{-1}}_{\text{coord. chart}} \in C^\infty(\varphi_\alpha(U_\alpha))$. Suppose $\underset{(\partial_i g)_{i=1\dots n}}{Dg}\Big|_{\varphi_\alpha(a)} = 0$. $\quad$ Let $h = f \circ \varphi_\beta^{-1}$

$g = h \circ \varphi_\beta \varphi_\alpha^{-1}$

$\Rightarrow g(x_1 \dots x_n) = h(y_1(x), \dots, y_n(x))$

$\Rightarrow \dfrac{\partial g}{\partial x_i} = \sum_j \dfrac{\partial h}{\partial y_j}(y(x)) \underbrace{\dfrac{\partial y_j}{\partial x_i}(x)}_{\substack{\text{invertible matrix,} \\ \text{since } y(x) \text{ is invertible}}}$ chain rule $\Rightarrow Dg|_{x(a)} = 0$ iff $Dh|_{y(a)} = 0$

$\Rightarrow$ vanishing of the derivative at a is independent of the coord. chart.

Let $Z_a = \{ f \in C^\infty(M) \mid f \text{ has vanishing derivative at } a \} \subset C^\infty(M)$ vect. subspace

def The cotangent space T_a^* at $a \in M$ is the quotient space $T_a^* = C^\infty(M)/Z_a$. The derivative of $f \in C^\infty(M)$ at $a \in M$ is its image in this space and is denoted $(df)_a$.

- if $f \in C^\infty(M)$, $(df)_a = d(\underset{\substack{\uparrow \\ \text{bump function} \equiv 1 \text{ in} \\ \text{the nbhd of } a}}{\mu} \cdot f)_a$ $\Rightarrow$ can make sense of $(df)_a$ for a locally-defined smooth f (in a nbhd of a), such as $f = x_1, \dots, x_n$ - loc. coord. functions

Proposition: Let M be an n-mfd. Then

- The cotangent space T_a^* at $a \in M$ is an n-dimensional vector space.
- If (U, φ) is a coord. chart around a with coords $x_1, \dots, x_n$, then the elements $(dx_1)_a, \dots, (dx_n)_a$ form a basis for T_a^*.
- If $f \in C^\infty(M)$ and in the coord. chart, $f \circ \varphi^{-1} = \psi(x_1, \dots, x_n)$ then

$$(df)_a = \sum_i \frac{\partial \psi}{\partial x_i}(\varphi(a)) (dx_i)_a \qquad (*)$$

Proof $f - \sum \dfrac{\partial \psi}{\partial x_i}(\varphi(a)) x_i$ - locally-defined smooth function whose derivative vanishes at a.

$\Rightarrow (df)_a = \sum \dfrac{\partial \psi}{\partial x_i}(\varphi(a)) (dx_i)_a$

and $(dx_i)_a$ span T_a^*. If $\sum_i \lambda_i (dx_i)_a = 0$ then $\sum_i \lambda_i x_i$ has vanishing derivative at $a \Rightarrow \lambda_1 = \dots = \lambda_n = 0$.

Rem We will denote $\underset{\substack{\uparrow \\ \text{coord. representation of } f}}{\psi} = f$.

So that $(*)$ becomes: $df = \sum_i \dfrac{\partial f}{\partial x_i} dx_i$.

With a change of coord. $(x_1, \dots, x_n) \longmapsto (y_1(x), \dots, y_n(x))$, we get

$$df = \sum_j \frac{\partial f}{\partial y_j} dy_j = \sum_{i,j} \frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial x_i} dx_i$$