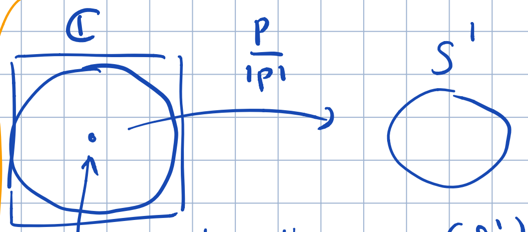
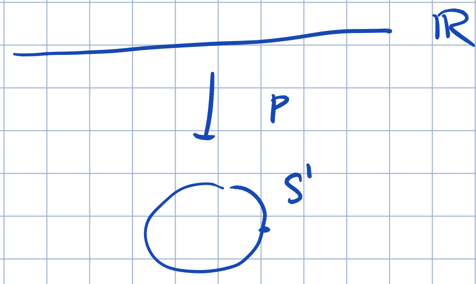
LAST TIME: $\pi_1(S^1) = \mathbb{Z}$



$r=0 \to$ triv elt in $\pi_1(S^1)$
$r=R \to$ nontriv. elt in $\pi_1(S^1)$

fund thm of algebra



$s \mapsto (\cos 2\pi s, \sin 2\pi s)$

Lemma (a)

For each path $\alpha : I \to S^1$ starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lift $\tilde{\alpha} : I \to \mathbb{R}$ starting at $\tilde{x}_0$.

i.e. $p \circ \tilde{\alpha} = \alpha$ or

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & S^1 \end{tikzcd}$$

Lemma (b)

For each homotopy $\alpha_t : I \to S^1$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lifted homotopy $\tilde{\alpha}_t : I \to \mathbb{R}$ of paths starting at $\tilde{x}_0$.

i.e. $p \circ \tilde{\alpha}_t = \alpha_t$

Lemma (c) ⟨homotopy lifting property⟩

Given a map $F : Y \times I \to S^1$ and a lifting

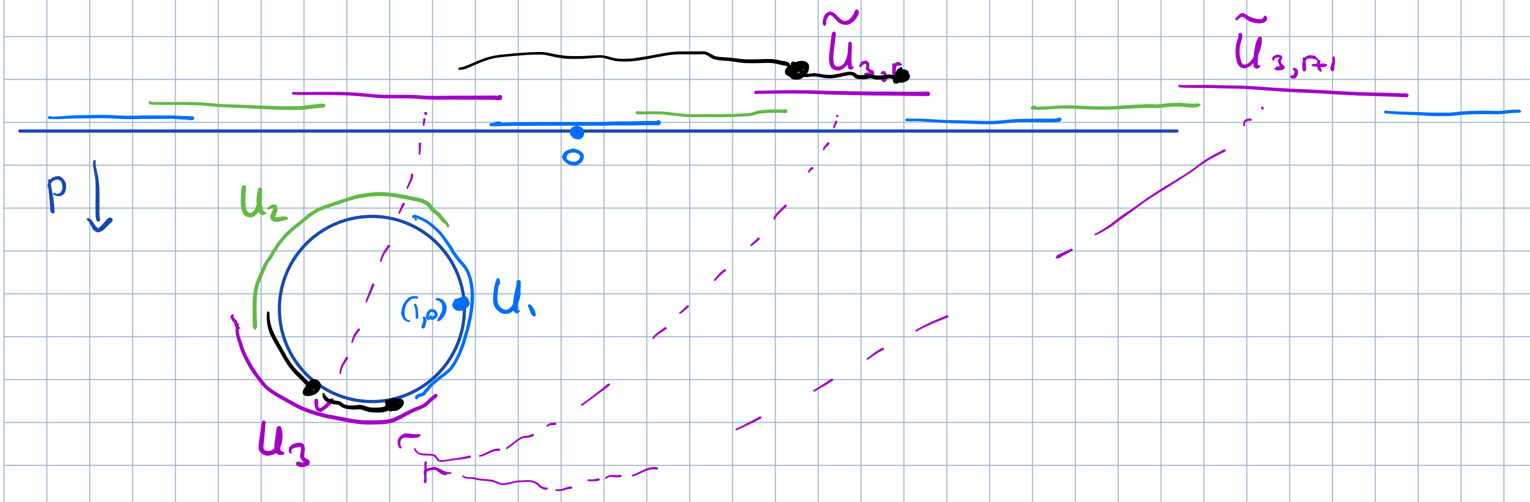
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ Y \times \{0\} \arrow[ur, dashed, "\tilde{F}_0"] \arrow[r, "F|_{Y\times\{0\}}"'] & S^1 \end{tikzcd},$$

there exists a unique lifting

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ Y \times I \arrow[ur, dashed, "\tilde{F}"] \arrow[r, "F"'] & S^1 \end{tikzcd}$$

restricting to given $\tilde{F}_0$ on $Y \times \{0\}$.

$Y = pt \Rightarrow (a)$ $\qquad Y = I \Rightarrow b$

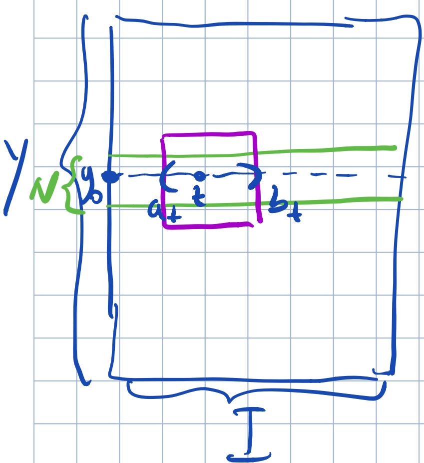


Can choose a cover of S^1 $\{U_\alpha\}$ s.t. $\forall \alpha$

$$p^{-1} U_\alpha = \coprod_r \underbrace{\tilde{U}_{\alpha,r}}_{\simeq U_\alpha}$$

U_α are "evenly covered"

construct $\tilde{F}$ locally in Y. Fix a pt $y_0 \in Y$



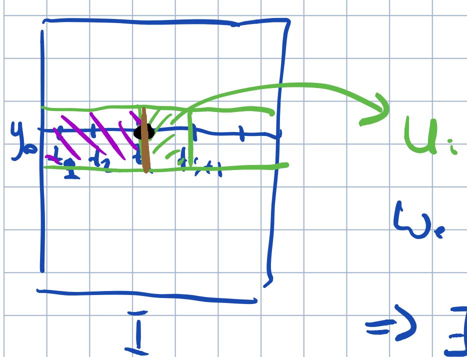
$\forall t$ (t, y_0) has a nbhd $\underset{\underset{y_0}{\psi}}{N_t} \times (a_t, b_t)$ $\subset Y \times I$

s.t. $F(N_t \times (a_t, b_t)) \subset U_\alpha$ for some α

$\{y_0\} \times I$ cpt $\Rightarrow$ fin. many (a_t, b_t) cover I

$\Rightarrow$ can choose $\underset{\underset{y_0}{\psi}}{N} \subset Y$ and a partition

$0 = t_0 < t_1 < t_2 < \dots < t_m = 1$ s.t. $F(N \times [t_i, t_{i+1}]) \subset U_{\alpha_i} = U_i$



Induction

Assume $\tilde{F}$ is constructed on $N \times [0, t_i]$.

We know that $F(N \times [t_i, t_{i+1}]) \subset U_i$

$\Rightarrow \exists\ \tilde{U}_{i,r} \subset \mathbb{R}$ containing the point $\tilde{F}(y_0, t_i)$

$$\begin{array}{c} \tilde{U}_{i,r} \\ {\scriptstyle p} \downarrow \wr\wr \\ U_i \end{array} \quad \Rightarrow \quad \tilde{F}(N \times \{t_i\}) \subset \tilde{U}_{i,r}$$

might need to replace
N with a smaller nbhd

with a smaller

$\Rightarrow$ define $\tilde{F}|_{N\times[t_i,t_{i+1}]} = p^{-1}\circ F|_{N\times[t_i,t_{i+1}]}$

$\tilde{U}_{i,r} \leftarrow U_i$

after fin. many repetitions we get $\tilde{F}|_{N\times I}$.

Uniqueness

$Y = pt$ Let $\tilde{F}, \tilde{F}'$ of $F: I \to S^1$ $\tilde{F}(0) = \tilde{F}'(0)$

as before $0 = t_0 < t_1 < \dots < t_m = 1$ s.t. $F([t_i, t_{i+1}]) \subset U_i$

Induction: assume $\tilde{F} = \tilde{F}'$ on $[0, t_i]$

$[t_i, t_{i+1}]$ connected $\Rightarrow$ $\tilde{F}([t_i, t_{i+1}])$ also connected $\Rightarrow$

$\Rightarrow \tilde{F}([t_i, t_{i+1}]) \subset \tilde{U}_{i,r}$ for some

$p \downarrow \wr$

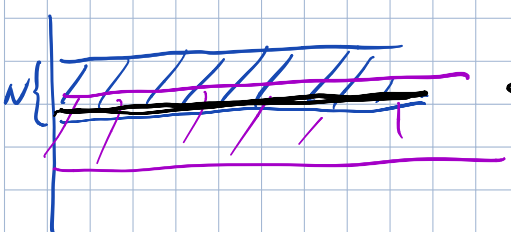
Similarly $\tilde{F}'([t_i, t_{i+1}]) \subset \tilde{U}_{i,r'}$ U_i

$r = r'$ because $\tilde{F}(t_i) = \tilde{F}'(t_i)$ in $\tilde{U}_{i,r}$ $\Rightarrow$ $r = r'$

$\Rightarrow$ since $p\circ\tilde{F} = p\circ\tilde{F}'$ on $[t_i, t_{i+1}]$ $\Rightarrow$ $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$

and $p: \tilde{U}_{i,r} \to U_i$

- proves uniqueness.



coincide on overlaps by uniqueness

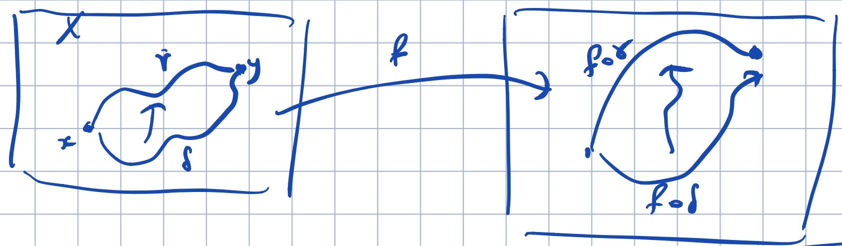
$\Rightarrow$ lift exists on Y and is unique.

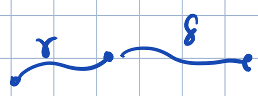
Induced map on π_1

$f: X \longrightarrow Y$ γ, δ path in X then

- If $\gamma(0) = \delta(0)$, $\gamma(1) = \delta(1)$ and $\gamma \sim \delta$ then

$f\circ\gamma \sim f\circ\delta$





$\gamma(1) = \delta(0)$ $\longrightarrow$ $\boxed{f \circ (\gamma * \delta) = (f \circ \gamma) * (f \circ \delta)}$

<u>def</u> Let $f : X \to Y$ ($x_0 \mapsto y_0$) a cont. map. The map (pushforward)

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$
$$[\alpha] \longmapsto [f \circ \alpha]$$

is called the map of fundamental groups induced by f.

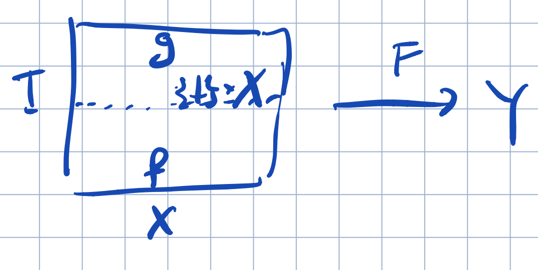
- $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ doesn't have to be surjective or injective
- if f is a homeo then f_* is an isomorphism

$$(f_*)^{-1} = (f^{-1})_*$$

Homotopy invariance of π_1

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$$

f, g are homotopic $f \sim g$ if $\exists$ a cont map $H : X \times I \to Y$ s.t. $H(x, 0) = f(x)$, $H(x, 1) = g(x)$



$f_t : X \to Y$

maps interpolating between f and g

- spaces X, Y are homotopic if

$$\begin{tikzcd} X \arrow[r, bend left, "\varphi"] & Y \arrow[l, bend left, "\psi"] \end{tikzcd} \quad \text{s.t.} \quad \psi \circ \varphi \sim id_X, \quad \varphi \circ \psi \sim id_Y$$

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if $\begin{tikzcd} (X, x_0) \arrow[r, bend left, "\varphi"] & (Y, y_0) \arrow[l, bend left, "\psi"] \end{tikzcd}$ – homotopies preserve x_0, y_0

then $\varphi_*: \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(Y, y_0)$.

Ex: $\pi_1(\mathbb{C} \setminus \{0\}) \simeq \pi_1(S^1) = \mathbb{Z}$.

$$\boxed{\bullet \quad \pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)}$$

Ex: $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$.