LAST TIME

Lemma (a)

For each path $\alpha : I \to S^1$ starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lift $\tilde{\alpha} : I \to \mathbb{R}$ starting at $\tilde{x}_0$.

i.e. $p \circ \tilde{\alpha} = \alpha$ or

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & S^1 \end{tikzcd}$$

Lemma (b)

For each homotopy $\alpha_t : I \to S^1$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lifted homotopy $\tilde{\alpha}_t : I \to \mathbb{R}$ of paths starting at $\tilde{x}_0$.

i.e. $p \circ \tilde{\alpha}_t = \alpha_t$

Lemma (c) <homotopy lifting property>

Given a map $F : Y \times I \to S^1$ and a lifting

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ Y \times \{0\} \arrow[ur, dashed, "\tilde{F}_0"] \arrow[r, "F|_{Y \times \{0\}}"'] & S^1 \end{tikzcd},$$

there exists a unique lifting

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ Y \times I \arrow[ur, dashed, "\tilde{F}"] \arrow[r, "F"'] & S^1 \end{tikzcd}$$

restricting to given $\tilde{F}_0$ on $Y \times \{0\}$.

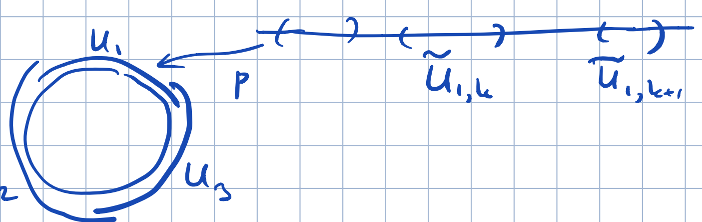
- taking $Y =$ point, we get Lm (a)
- taking $Y = I$, we get Lm (b)

Proof of Lm (c) Use that $\exists$ an open cover $\{U_\alpha\}$ of S^1 s.t. $\forall \alpha$ $p^{-1}(U_\alpha) \overset{(*)}{\simeq} \coprod_j \tilde{U}_{\alpha, j}$

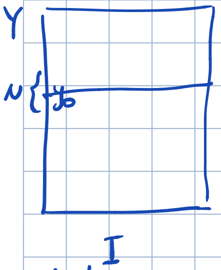
$$\begin{tikzcd} \tilde{U}_{\alpha,j} \arrow[d, "p", "\cong"'] \\ U_\alpha \end{tikzcd}$$

(U_α's are "evenly covered" by p)

Ex:



- construct the lift locally on Y, i.e. a nbhd $N \subset Y$, $N \ni y_0$



(t, y_0) has a nbhd $N_t \times (a_t, b_t) \subset Y \times I$ s.t. $F(N_t \times (a_t, b_t)) \subset U_\alpha$ for some α

$\{y_0\} \times I$ cpt $\Rightarrow$ fin. many (a_t, b_t)'s cover $\{y_0\} \times I$

$\Rightarrow$ can choose $N \subset Y$ ($\ni y_0$) and a partition $0 = t_0 < t_1 < \dots < t_m = 1$

s.t. $F(N \times [t_i, t_{i+1}]) \subset U_{\alpha_i} =: U_i$

Induction: Assume $\tilde{F}$ is constructed on $N \times [0, t_i]$. We know that $F(N \times [t_i, t_{i+1}]) \subset U_i$

(*) $\Rightarrow \exists\, \tilde{U}_{i,r} \subset \mathbb{R}$ containing the point $\tilde{F}(y_0, t_i)$ $\Rightarrow$ $\tilde{F}(N \times \{t_i\}) \subset \tilde{U}_{i,r}$

$\tilde{U}_{i,r} \xrightarrow[\cong]{p} U_i$

replacing N with a smaller nbhd

$\Rightarrow$ define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ as $p^{-1} F|_{N \times [t_i, t_{i+1}]}$. ($\tilde{U}_{i,r} \xleftarrow{\sim} U_i$)

After finitely many repetitions, we get $\tilde{F}|_{Y \times I}$ ✓.

Uniqueness of the lift for $Y = pt$: Let $\tilde{F}, \tilde{F}'$ be two lifts of $F: I \to S^1$, $\tilde{F}(0) = \tilde{F}'(0)$.

as before: $0 = t_0 < t_1 < \dots < t_m = 1$ s.t. $F([t_i, t_{i+1}]) \subset U_i$ some

Induction: assume $\tilde{F} = \tilde{F}'$ on $[0, t_i]$.

$[t_i, t_{i+1}]$ connected $\Rightarrow$ $\tilde{F}([t_i, t_{i+1}])$ connected $\Rightarrow$ $\tilde{F}([t_i, t_{i+1}]) \subset \tilde{U}_{i,r}$ for some r ($\tilde{U}_{i,r} \xrightarrow[\cong]{p} U_i$)

similarly $\tilde{F}'([t_i, t_{i+1}]) \in \tilde{U}_{i,r'}$

$\tilde{F}(t_i) = \tilde{F}'(t_i) \Rightarrow r = r' \Rightarrow \tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$ ✓

since $p\tilde{F} = p\tilde{F}'$ and p injective on $\tilde{U}_{i,r}$

- $\tilde{F}$ constructed on $N \times I$ are unique when restricted to $\{y\} \times I$ $\Rightarrow$ must agree when $N \times I$ and $N' \times I$ overlap

$\Rightarrow$ we have a well-defined lift $\tilde{F}$ on $Y \times I$. ☐

Induced map on π_1

- Let $f: X \to Y$ cont. map, γ, δ - paths in X. Then:
  - if γ and δ have same endpoints and $\gamma \simeq \delta$, then $f \circ \gamma \simeq f \circ \delta$ — Compatibility with homotopies
  - if $\gamma(1) = \delta(0)$, then $f \circ (\gamma * \delta) = (f \circ \gamma) * (f \circ \delta)$ — compat. with $*$.

def: Let $f: X \to Y$, $x_0 \mapsto y_0$ cont. map. The map $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $[\gamma] \mapsto [f \circ \gamma]$ is called the map of fund. groups induced by f.

- $f_*: \pi_1(X,x_0) \to \pi_1(Y,y_0)$ doesn't have to be injective or surjective for general f.

didn't cover

- if $(X,x_0) \xrightarrow{f} (Y,y_0) \xrightarrow{g} (Z,z_0)$, then $(g \circ f)_* = g_* \circ f_*$

- if f is a homeo, then f_* is an isomorphism.

- $\pi_1(X \times Y, (x_0,y_0))$ is isomorphic to $\pi_1(X,x_0) \times \pi_1(Y,y_0)$

<homework>

didn't cover

Idea: $\left(\text{a loop in } X\times Y \quad I \xrightarrow{\gamma} X \times Y\right) \longmapsto \left(I \xrightarrow{p_1\circ\gamma} X,\ I \xrightarrow{p_2 \circ \gamma} Y\right)$ – component loops

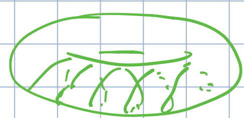
$$\begin{array}{ccc} & X\times Y & \\ {}^{p_1}\swarrow & & \searrow{}^{p_2} \\ X & & Y \end{array}$$

Ex: $\pi_1(S^1 \times S^1) \simeq \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.

torus

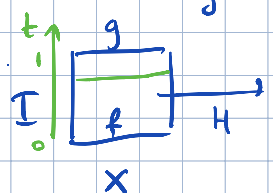
$[\omega_{pq}(s) = (\omega_p(s), \omega_q(s))] \longleftarrow (p,q)$

torus winding



Homotopy invariance of π_1

- two cont. maps $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ are called "homotopic" $f \sim g$ if $\exists$ a cont. map $H: X \times I \to Y$ s.t. $H(x,0) = f(x)$, $H(x,1) = g(x)$



= "interpolating family of maps" $f_t: X \to Y$

- two top spaces X, Y are called homotopy equivalent if $\exists\ \varphi, \psi$ $\quad X \overset{\varphi}{\underset{\psi}{\rightleftarrows}} Y$ s.t. $\psi \circ \varphi \sim id_X$, $\varphi \circ \psi \sim id_Y$

Ex: $X \times I \overset{\varphi}{\underset{\psi}{\rightleftarrows}} X$

$\varphi: (x,s) \longmapsto x$
$\psi: x \longmapsto (x,0)$

$\varphi \circ \psi = id_X$
$\psi \circ \varphi: (x,s) \longmapsto (x,0)$

homotopy $\psi \circ \varphi \overset{H_t}{\sim} id_{X \times I}$: $\quad H_t: (x,s) \longmapsto (x, ts)$
$0 \le t \le 1$

didn't cover

Ex: $\begin{tikzcd} \mathbb{C}\setminus\{0\} \arrow[r, bend left, "\varphi"] & S^1 \arrow[l, bend left, "\psi"] \end{tikzcd}$ ← inclusion $\varphi: z \longmapsto \frac{z}{|z|}$

we have $\varphi \circ \psi = id_{S^1}$, $\underset{z \mapsto \frac{z}{|z|}}{\psi \circ \varphi} \sim id_{\mathbb{C}\setminus\{0\}}$

with $H(t, z) = |z|^t \frac{z}{|z|}$

Lemma: if $\begin{tikzcd}(X,x_0) \arrow[r, bend left, "\varphi"] & (Y,y_0) \arrow[l, bend left, "\psi"]\end{tikzcd}$ homotopy equivalence of pointed top spaces (i.e. homotopies preserve base points)

then $\varphi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is an isomorphism. ← <homework>

Ex: $\pi_1(\mathbb{C}\setminus\{0\}) \simeq \pi_1(S^1) \cong \mathbb{Z}$

π_1 as a functor

π_1 is a functor from the category of pointed top. spaces to the category of groups

def a category $\mathcal{C}$ is:

DATA
- a class $Ob(\mathcal{C})$ of "objects"
- for $x, y \in Ob(\mathcal{C})$, a set $Mor_{\mathcal{C}}(x,y)$ of "morphisms" from x to y
- identity morphism $id_x \in Mor_{\mathcal{C}}(x,x)$ for each $x \in Ob(\mathcal{C})$
- composition rule $Mor_{\mathcal{C}}(x,y) \times Mor_{\mathcal{C}}(y,z) \longrightarrow Mor_{\mathcal{C}}(x,z)$

$$\left(x \xrightarrow{f} y,\ y \xrightarrow{g} z\right) \longmapsto \left(x \xrightarrow{g \circ f} z\right)$$

Axioms:

- composition is associative: for $w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$, $\boxed{h \circ (g \circ f) = (h \circ g) \circ f}$

- identity property: for $x \xrightarrow{f} y$, $\boxed{f \circ id_x = id_y \circ f = f}$

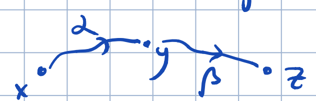
- a morphism $x \xrightarrow{f} y$ in $\mathcal{C}$ is called an isomorphism if $\exists\ y \xrightarrow{g} x$ (inverse) s.t. $g \circ f = id_x$, $f \circ g = id_y$

Examples

Category C	Set	Grp	Vect	Top	Top$_*$
objects	sets	groups	vector spaces	top. spaces	pointed top. spaces
morphisms	maps	homomorphisms	linear maps	continuous maps	cont. maps s.t. $x_0 \mapsto y_0$
isomorphisms	bijections	group isomorphisms	linear isomorphisms	homeomorphisms	homeo preserving basepoints

More examples

- for G a group, we can construct a category C with a single object $*$, with $\mathrm{Mor}(*,*) = G$ (note: all morphisms are isomorphisms!)
  $\circ$ = group product
- for X a top space, can construct a category $\Pi_1(X)$ - "fundamental groupoid".
  objects = points of X.
  $\mathrm{Mor}(x,y) = $ paths from x to y / homotopy
  composition: $[\beta]\circ[\alpha] := [\alpha * \beta]$



def Let C, D be categories. A functor $F: C \to D$

- associates to every object x of C an object $F(x)$ of D.
- associates to every morphism $f \in \mathrm{Mor}_C(x,y)$ a morphism $F(f) \in \mathrm{Mor}_D(F(x), F(y))$,

So that:

- $F(g\circ f) = F(g)\circ F(f)$ - compatibility with compositions
- $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ - compatibility with identities

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- $\boxed{\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Grp}}$ is a functor!

on objects: $(X, x_0) \longmapsto \pi_1(X, x_0)$

on morphisms: $\begin{pmatrix} f: X \to Y \\ x_0 \mapsto y_0 \end{pmatrix} \longmapsto \left(f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0) \right)$