9/7/20

• $\pi_1(X \times Y, (x_0, y_0)) \overset{\text{isomorphic to}}{\simeq} \pi_1(X, x_0) \times \pi_1(Y, y_0)$

Why? - $\overset{\text{a loop}}{\gamma} : I \longrightarrow X \times Y$ is a pair (loop γ^1 in X, loop γ^2 in Y)

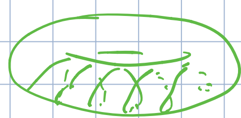
$s \longmapsto (\gamma^1(s), \gamma^2(s))$

Similarly, a homotopy γ_t of loops in $X \times Y$ is a pair of homotopies.

Ex: $\underset{\text{torus}}{\pi_1(S^1 \times S^1)} \simeq \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.

$[\omega_{pq}(s) = (\omega_p(s), \omega_q(s))] \longleftarrow\!\shortmid (p, q)$

torus winding



• induced maps of π_1 satisfy: if $X \xrightarrow{f} Y \xrightarrow{g} Z$

then $(g \circ f)_* = g_* \circ f_* : \pi_1(X) \longrightarrow \pi_1(Z)$.

π_1 as a functor

π_1 is a functor from the category of pointed top. spaces to the category of groups

def a category $\mathcal{C}$ is:

DATA

- a class $Ob(\mathcal{C})$ of "objects"
- for $x, y \in Ob(\mathcal{C})$, a set $Mor_{\mathcal{C}}(x,y)$ of "morphisms" from x to y
- identity morphism $id_x \in Mor_{\mathcal{C}}(x,x)$ for each $x \in Ob(\mathcal{C})$
- composition rule $Mor_{\mathcal{C}}(x,y) \times Mor_{\mathcal{C}}(y,z) \longrightarrow Mor_{\mathcal{C}}(x,z)$

$$(x \xrightarrow{f} y, \; y \xrightarrow{g} z) \longmapsto (x \xrightarrow{g \circ f} z)$$

Axioms:

- composition is associative: for $w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$, $\boxed{h \circ (g \circ f) = (h \circ g) \circ f}$

- identity property: for $x \xrightarrow{f} y$, $\boxed{f \circ id_x = id_y \circ f = f}$

- a morphism $x \xrightarrow{f} y$ in C is called an isomorphism if $\exists\ y \xrightarrow{g} x$ (inverse) s.t. $g \circ f = id_X$, $f \circ g = id_Y$

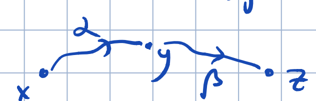
Examples

Category C	Set	Grp	Vect	Top	Top*
objects	sets	groups	vector spaces	top. spaces	pointed top. spaces
morphisms	maps	homomorphisms	linear maps	continuous maps	cont. maps s.t. $x_0 \mapsto y_0$
isomorphisms	bijections	group isomorphisms	linear isomorphisms	homeomorphisms	homeo preserving base points

More examples

- for G a group, we can construct a category C with a single object $*$, with $Mor(*,*) = G$ (note: all morphisms are isomorphisms!)
  $\circ$ = group product

- for X a top space, can construct a category $\Pi_1(X)$ - "fundamental groupoid".
  objects = points of X.
  $Mor(x,y)$ = paths from x to y / homotopy
  composition: $[\beta] \circ [\alpha] := [\alpha * \beta]$



def Let C, D be categories. A functor $F: C \to D$

- associates to every object x of C an object $F(x)$ of D.
- associates to every morphism $f \in Mor_C(x,y)$ a morphism $F(f) \in Mor_D(F(x), F(y))$,

So that:

(a) $F(g \circ f) = F(g) \circ F(f)$ - compatibility with compositions

(b) $F(id_x) = id_{F(x)}$ - compatibility with identities

- $\boxed{\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Grp}}$ is a functor!

on objects: $(X, x_0) \longmapsto \pi_1(X, x_0)$

on morphisms: $\left(\begin{array}{c} f: X \to Y \\ x_0 \mapsto y_0 \end{array}\right) \longmapsto \left(\begin{array}{c} f_*: \pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0) \\ [\gamma] \longmapsto [f\circ\gamma] \end{array}\right)$

functoriality: $(g\circ f)_* = g_* \circ f_*$, $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X)}$

Seifert - van Kampen theorem

- main tool for computing π_1 :

Assume $X = X_1 \cup X_2$ with $X_1 \cap X_2 = Y$, Y is path connected, $x_0 \in Y$,

open subspaces of X

let $j_a : Y \longrightarrow X_a$, $a = 1, 2$.

Then $\pi_1(X, x_0) = \boxed{\pi_1(X_1, x_0) \underset{\pi_1(Y,x_0)}{*} \pi_1(X_2, x_0)}$

free product of groups $\pi_1(X_1), \pi_1(X_2)$ amalgamated over $\pi_1(Y)$ w.r.t. maps $(j_1)_*: \pi_1(Y) \to \pi_1(X_1)$, $(j_2)_*: \pi_1(Y) \to \pi_1(X_2)$

- **Free product of groups**

Let G_1, G_2 be two groups. Their free product $G_1 * G_2$ is the group where elements are equivalence classes of words $g_1 \cdots g_k$ with $g_i \in G_1$ or G_2 (we assume G_1 and G_2 are disjoint as sets),

with equivalence rel. $\sim$ generated by

(i) $g_1 \cdots g_i \cdots g_k \sim g_1 \cdots \hat{g_i} \cdots g_k$ if $g_i = 1_{G_1}$ or 1_{G_2}

(ii) $g_1 \cdots g_i g_{i+1} \cdots g_k \sim g_1 \cdots (g_i \cdot g_{i+1}) \cdots g_k$ if g_i, g_{i+1} both in G_1 or both in G_2

product in G_1 or G_2.

multiplication on $G_1 * G_2$ = concatenation of words.

identity = "empty word"

inverse: $g_1 \cdots g_k \longmapsto g_k^{-1} \cdots g_1^{-1}$.

- One has group homomorphisms $G_1 \xrightarrow{i_1} G_1 * G_2 \xleftarrow{i_2} G_2$

mapping $g_{\in G_a}$ to a 1-letter word

"reduced words"

Ex: $\mathbb{Z} * \mathbb{Z} = \dfrac{\{\text{words in } a, a^{-1}, b, b^{-1}\}}{\sim} = \left\{ x_1^{n_1} \cdots x_k^{n_k} \;\middle|\; \begin{array}{l} n_k \neq 0 \\ \text{either } x_1 = a,\, x_2 = b,\, x_3 = a,\, x_4 = b \text{ etc} \\ \text{or } x_1 = b,\, x_2 = a \text{ etc.} \end{array} \right\} \cup \{1\}$

$\{\ldots, a^{-1}, 1, a, a^2, \ldots\}$ $\quad$ $\{\ldots, b^{-1}, 1, b, b^2, \ldots\}$

$= \langle a, b \rangle$ – "free group with two generators a, b"

- $\mathbb{Z}_2 * \mathbb{Z}_2 = \,?$ $\quad$ e.g. is it a finite group?

$\{1, a\} = \langle a \rangle / a^2 = \langle a \mid a^2 = 1 \rangle$ $\quad$ $\mathbb{Z}_2 = \langle b \mid b^2 = 1 \rangle$

quotient by normal subgroup generated by a^2 $\quad$ generator $\quad$ relation

$\mathbb{Z}_2 * \mathbb{Z}_2 = \langle \underbrace{a, b}_{\text{generators}} \mid \underbrace{a^2 = 1,\ b^2 = 1}_{\text{relations}} \rangle$

- Universal property: if $G_1 \xrightarrow{f_1} H$, $G_2 \xrightarrow{f_2} H$ are group homomorphisms (for some H),

then $\exists!$ group hom $f : G_1 * G_2 \longrightarrow H$ st

diagram
$$\begin{tikzcd} G_1 \arrow[d, "i_1"'] \arrow[dr, "f_1"] & \\ G_1 * G_2 \arrow[r, dashed, "f"] & H \\ G_2 \arrow[u, "i_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$
is commutative: $f_1 = f \circ i_1$, $f_2 = f \circ i_2$

<proof → homework>

Amalgamated free product.

def Let G_1, G_2, A be groups, $j_1 : A \to G_1$, $j_2 : A \to G_2$ homomorphisms.

Then: $G_1 \underset{A}{*} G_2 := G_1 * G_2 \big/ \begin{array}{l}\text{normal subgroup} \\ \text{generated by elements} \\ j_1(\alpha) j_2(\alpha)^{-1} \ \forall \alpha \in A\end{array} = \dfrac{\text{words of elements of } G_1, G_2}{\begin{array}{c}\sim \text{ as before, plus} \\ \ldots j_1(\alpha) \ldots \sim \ldots j_2(\alpha) \ldots \\ \forall \alpha \in A\end{array}}$

"amalgamated free product"

Ex: • $G_1 \underset{1}{*} G_2 = G_1 * G$ - usual free product

1 – triv. group

- $\langle a_1, \ldots, a_k \mid \text{relations}_1 \rangle \underset{A}{*} \langle b_1, \ldots, b_\ell \mid \text{relation}_2 \rangle$

$= \langle a_1, \ldots a_k, b_1, \ldots, b_\ell \mid \text{relations}_1, \text{relations}_2, \{ j_1(\alpha) = j_2(\alpha) \}_{\alpha \in A} \rangle$

• $\underset{\langle a\rangle}{\mathbb{Z}} \underset{\underset{\langle b\rangle}{\mathbb{Z}}}{*} 1 = ?$ if $j_1 : b \mapsto a^k$, $j_2 : b \mapsto 1$

$= \langle a \mid a^k = 1 \rangle = \{1, a, a^2, \dots, a^{k-1}\} \simeq \mathbb{Z}/_k\mathbb{Z}$ the group of residues mod. k.

- notice that it does depend on the maps j_a! (via k)

• $\underset{\langle a,b\rangle}{(\mathbb{Z}*\mathbb{Z})} \underset{\underset{\langle c\rangle}{\mathbb{Z}}}{*} 1$ $c \xmapsto{j_1} aba^{-1}b^{-1}$, $c \mapsto 1$

$= \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$

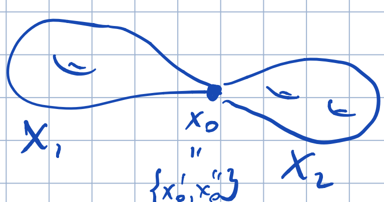
or: $ab = ba$

Using Seifert-van Kampen to compute $\pi_1(X)$ - examples

Recall SvK: (#) $\boxed{\pi_1(X) = \pi_1(X_1) \underset{\pi_1(Y)}{*} \pi_1(X_2)}$ w.r.t. $(j_a)_*$ if $X = X_1 \cup X_2$ open, $Y = X_1 \cap X_2$ - path connected, contains x_0, $j_a : Y \hookrightarrow X_a$

Ex: (1) $X = X_1 \vee X_2$ - "wedge sum of (pointed) spaces $\underset{x_0'}{X_1}$, $\underset{x_0''}{X_2}$" ($x_0' \in X_1$, $x_0'' \in X_2$)

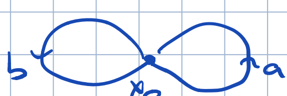
$= X_1 \amalg X_2 / _{x_0' \sim x_0''}$



By (#): $\pi_1(X, x_0) = \pi_1(X_1, x_0') * \pi_1(X_2, x_0'')$

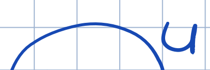
- free product

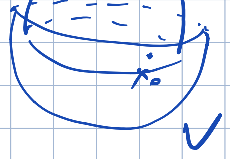
(1') E.g. $\pi_1(S^1 \vee S^1, x_0) = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$



(2) $\pi_1(S^2) = \pi_1(U) \underset{\pi_1(U \cap V)}{*} \pi_1(V) = 1 \underset{\mathbb{Z}}{*} 1 = 1$ - trivial!

$U \approx$ open disk, $U \cap V \approx S^1 \times (-1, 1)$





$U: 0 \le \theta < \frac{\pi}{2} + \varepsilon$, $\quad V: \frac{\pi}{2} - \varepsilon < \theta \le \pi$ ⑥

azimuthal angle

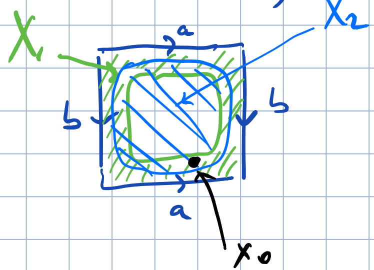
$W = U \cap V : \frac{\pi}{2} - \varepsilon < \theta < \frac{\pi}{2} + \varepsilon$

Similarly: $\pi_1(S^n) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = 1 * 1 = 1$, $n \geq 2$

extended upper hemisphere $\approx$ open disk

1 or $\mathbb{Z}$: $n \geq 3$, $n = 2$

(3) $\pi_1(\text{torus})$ - via van Kampen (already know it as a π_1 of a product)



$T = X_1 \cup X_2$, $\quad Y = X_1 \cap X_2 \approx S^1 \times (-1,1) \sim S^1$

X_1: nbhd of the based loops a, b $\sim$ wedge of two circles

X_2: open disk $\sim$ pt

So: $\pi_1(T) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) = (\mathbb{Z} * \mathbb{Z}) *_{\mathbb{Z}} 1$ ⇔

$\mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$, $\quad \mathbb{Z} = \langle c \rangle$

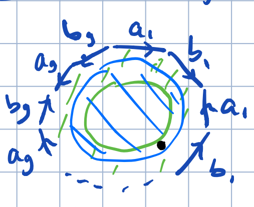
with $c \xrightarrow{(j_1)_*} aba^{-1}b^{-1} \in \pi_1(X_1)$

$c \xrightarrow{(j_2)_*} 1 \in \pi_1(X_2)$

⇔ $\langle a, b \mid aba^{-1}b^{-1} = 1 \rangle = \mathbb{Z}^2$.

(3') Similarly: $\pi_1(\Sigma_g) = \mathbb{Z}^{*2g} *_{\mathbb{Z}} 1 = \boxed{\langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle}$

$\mathbb{Z}^{*2g} = \langle a_1, b_1, \ldots, a_g, b_g \rangle$, $\quad \mathbb{Z} = \langle c \rangle$



4g-gon

with $c \xrightarrow{j_1} a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

$c \xrightarrow{j_2} 1$