BASIC GEOMETRY AND TOPOLOGY HOMEWORK 5, DUE 9/18/2020

I Prove that the fundamental group of an n-dimensional real projective space is

$$\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$$

for $n \geq 2$. (Hint: use Seifert-van Kampen theorem and induction in n.)

II Let Γ be a rooted tree (a connected graph with no cycles and a distinguished vertex – the "root"); let us regard Γ as an oriented graph, with edges oriented *toward* the root. Consider the category C_Γ where the objects are vertices of Γ and the morphisms from vertex x to vertex y are the directed paths along edges of Γ from x to y (plus the trivial path from x to x as the identity morphism).[1] Does the coproduct in C_Γ exist for every pair of objects? If yes, describe it.

III (a) Prove that the free product is the coproduct in the category of groups. I.e., prove the universal property of the free product of groups: let G_1, G_2 be two groups and $i_1 : G_1 \to G_1 * G_2$, $i_2 : G_2 \to G_1 * G_2$ the inclusions of $G_{1,2}$ into the free product as 1-letter words. Let H be another group and $f_1 : G_1 \to H$, $f_2 : G_2 \to H$ two homomorphisms. Prove that then there exists a unique homomorphism $f : G_1 * G_2 \to H$ such that $f \circ i_1 = f_1$ and $f \circ i_2 = f_2$.

(b) Prove that the wedge sum $X_1 \vee X_2$ is the coproduct in the category of pointed topological spaces.

(c) Let $X = X_1 \cup X_2$ be a topological space presented as a union of two open subsets. Show that, if $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are two continuous maps to another topological space Y, agreeing on $X_1 \cap X_2$, then there exists a unique *continuous* map $f : X \to Y$ restricting to f_1 on X_1 and restricting to f_2 on X_2. In other words, prove that X is the pushout (in the category Top) of the diagram $X_1 \xleftarrow{i_1} X_1 \cap X_2 \xrightarrow{i_2} X_2$ where i_1, i_2 are the inclusions.

IV Let S^1 be the unit circle in $\mathbb{C}$ and consider the map $f : S^1 \to S^1$, $z \to z^n$ (for n some integer). Compute the induced map $f_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$.[2]

V (a) Compute the fundamental group of $\mathbb{R}^2$ with n distinct points removed.

(b) Compute the fundamental group of S^2 with n distinct points removed.

[1] Note that there is at most one path from x to y, since Γ is a tree.

[2] Recall that $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$ with the isomorphism given by the winding number of the loop.