BASIC GEOMETRY AND TOPOLOGY HOMEWORK 6, DUE 10/2/2020

I Let X be a path-connected reasonable (i.e. locally path-connected and semilocally simply connected) space, with $x_0 \in X$ a base point. Consider the action of $\pi_1(X, x_0)$ on the universal covering $\widetilde{X}$ given by $[\alpha] : [\gamma] \mapsto [\alpha * \gamma]$ with α a based loop in (X, x_0) representing an element of π_1 and γ a path in X starting at x_0, so that $[\gamma]$ is a point of $\widetilde{X}$. Prove that this action of π_1 on $\widetilde{X}$ is free and its orbits are precisely the fibers of the covering map $p : \widetilde{X} \to X$, $[\gamma] \mapsto \gamma(1)$.

II (a) Prove that if $p : \widetilde{X} \to X$ is an N-sheeted covering[1] of a path-connected finite CW complex X, then $\widetilde{X}$ also inherits the structure of a CW complex from X. Prove that the Euler characteristics satisfy the relation

$$\chi(\widetilde{X}) = N \cdot \chi(X)$$

(b) Prove that if one has an N-sheeted covering $p : \Sigma_g \to \Sigma_h$ of the surface of genus h by the surface of genus g, then one has

$$g - 1 = N \cdot (h - 1)$$

In particular, if $h \neq 1$, the ratio $\frac{g-1}{h-1}$ must be a positive integer.

III Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a (possibly non-normal) based covering over a reasonable path-connected space X. Show that for an element $[\gamma] \in \pi_1(X, x_0)$, the action of $[\gamma]$ on $\widetilde{X}$ by a deck trasformation $\rho([\gamma]) : (\widetilde{X}, \widetilde{x}_0) \to (\widetilde{X}, \widetilde{x}_1)$ given by lifting the map p along the covering $p' : (\widetilde{X}, \widetilde{x}_1) \to (X, x_0)$ is well-defined if and only if $[\gamma]$ belongs to the *normalizer* $N(H)$ of $H = p_* \pi_1(\widetilde{X}, \widetilde{x}_0)$, i.e. iff $[\gamma]^{-1} H [\gamma] = H$. Here $\widetilde{x}_1 = \widetilde{\gamma}(1)$ – the endpoint of the lifting of the path γ to $\widetilde{X}$, starting at $\widetilde{x}_0$.

IV Give an example of a non-normal covering with a nontrivial group of deck transformations. (E.g., you can think about coverings of the wedge of two circles.)

V Consider the sphere S^n regarded as the unit sphere in $\mathbb{R}^{n+1}$ with an atlas consisting of two charts

$$\begin{array}{ccc} U^{\pm} = S^n \backslash \{(0, \ldots, 0, \pm 1)\} & \xrightarrow{\phi^{\pm}} & \mathbb{R}^n \\ (x_0, \ldots, x_n) & \mapsto & \frac{1}{1 \mp x_n}(x_0, \ldots, x_{n-1}) \end{array}$$

Calculate the transition map $\phi^- \circ (\phi^+)^{-1}$. What is its domain and image?

[1] We are assuming that N is finite.