BASIC GEOMETRY AND TOPOLOGY HOMEWORK 9, DUE 10/23/2020

I (a) Consider the vector field $X = (1-x^2)\frac{\partial}{\partial x}$ on the open interval $M = (-1, 1)$ Construct explicitly (by solving the ODE) the maximal integral curve of X passing at time $t = 0$ through a point $a \in M$. Does the global flow of X exist? (I.e., as a map $\phi : \mathbb{R} \times M \to M$.) If yes, find it explicitly; otherwise, write find the maximal non-global flow and its domain $U \subset \mathbb{R} \times M$.

(b) Same questions as in (a), for the vector field $Y = \frac{\partial}{\partial x}$ on the same manifold $M = (-1, 1)$.

II Conseider the vector fields X, Y from the Problem I and compute their Lie bracket in two ways:

(a) Calculate the Lie bracket of X and Y as[1]

$$[X,Y]_a = -\left.\frac{d}{dt}\right|_{t=0} \left(D\phi^Y_{-t}\right)_{\phi^Y_t(a)} X_{\phi^Y_t(a)}$$

(b) Compare the result of (c) with the direct computation of the Lie bracket $[X, Y] = X \circ Y - Y \circ X$.

III Let V be an n-dimensional real vector space. The *symmetric algebra* $S^\bullet V$ of V is defined as the quotient of the tensor algebra TV by the ideal generated by elements of the form $v \otimes w - w \otimes v$ with $v, w \in V$. Let $S^p V := \pi(V^{\otimes p})$ be the *p-th symmetric power of* V, where $\pi : TV \to S^\bullet V$ is the quotient map.

(a) Prove that the product in $S^\bullet V$ is commutative: for $\alpha \in S^p V$, $\beta \in S^q V$, one has $\alpha\beta = \beta\alpha \quad \in S^{p+q}V$.[2]

(b) Prove that if $v_1, \ldots, v_n$ is a basis in V then the set of vectors $\{v_{i_1} \cdots v_{i_p}\}_{1 \le i_1 \le \cdots \le i_p \le n}$ forms a basis in $S^p V$.

(c) Find the dimension of $S^p V$ as a real vector space.

IV Consider the triple of flows ϕ^i_t on $\mathbb{R}^3$ given by rotation by the angle t about the coordinate axis x_i, with $i = 1, 2, 3$.[3]

(a) Note that one has an action of the group $SO(3)$ on $\mathbb{R}^3$ (by matrix-vector multiplication) and flows ϕ^i_t correspond to three special 1-dimensional subgroups in $SO(3)$; identify these subgroups.

(b) For each flow ϕ^i_t, find the corresponding vector field R_i on $\mathbb{R}^3$.

(c) Prove that we have the following Lie brackets:

$$[R_1, R_2] = -R_3, \quad [R_2, R_3] = -R_1, \quad [R_3, R_1] = -R_2$$

[1] This formula is equivalent to the "geometric" formula for the Lie bracket of vector fields $[X, Y]$ given in the class: the roles of X, Y are interchanged and the total sign is changed.

[2] The product in TV is defines as induced from the product in TV, i.e., if $\alpha = \pi(a)$, $\beta = \pi(b)$, then $\alpha\beta = \pi(a \otimes b)$.

[3] We understand that the rotation is counterclockwise if seen from the positive direction of the axis.

V (a) Let V, W, U be finite-dimensional vector spaces and $\mathbb{B} : V \times W \to U$ a bilinear map. Let ϕ be the map from U^* to bilinear forms on $V \times W$ given by $\phi(\xi) = \xi \circ \mathbb{B}$. Let $\beta : V \otimes W \to U$ be the dual map to ϕ.[4] Show that then one has $\beta(v \otimes w) = \mathbb{B}(v, w)$, i.e., β is the map making the universal property of the tensor product work for $U \otimes V$ defined as the dual of the space of bilinear forms on $V \times W$.

(b) The general construction of the tensor product (not requiring the vector spaces to be finite-dimensional) is as the quotient space

$$V \otimes W := F(V \times W)/\sim$$

where

$$F(V \times W) = \{\sum_i c_i(v_i, w_i) \mid c_i \in \mathbb{R}, v_i \in V, w_i \in W\}$$

– formal sums of pairs of vectors from V, W with real coefficients, where only *finite sums* are allowed.[5] The equivalence relation $\sim$ is generated by

$$(v, w)+(v', w) \sim (v+v', w), \quad (v, w)+(v, w') \sim (v, w+w'), \quad c(v, w) \sim (cv, w) \sim (v, cw)$$

The tensor product $v \otimes w$ of two vectors $v \in V$, $w \in W$ is defined as the equivalence class of the pair (v, w). Prove that for finite-dimensional vector spaces this construction is equivalent to the one given in class (the dual space to the space of bilinear forms on $V \times W$).

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[4] More precisely: the dual map to ϕ goes to $(U^*)^*$ and we compose it with the canonical isomorphism $(U^*)^* \to U$ (the inverse of the canonical inclusion $U \to (U^*)^*$, $u \mapsto (\xi \in U^* \mapsto \xi(u))$ which for U finite-dimensional is an isomorphism) to obtain β.

[5] $F(V \times W)$ is called the "free vector space" on the set $V \times W$.