Mathai - Quillen representative for Thom & Euler classes

Ref: E. Getzler "The Thom class of Mathai and Quillen and probability theory"

- Berezin integral

$$\int^{\text{Berezin}} : \mathbb{R}\langle\theta_1,\dots,\theta_n\rangle / \theta_i\theta_j = -\theta_j\theta_i \longrightarrow \mathbb{R}$$

$$f = \sum_{k=0}^{n} \sum_{1\le i_1<\dots<i_k\le n} \underbrace{f_{i_1\dots i_k}}_{\text{coefficients}} \theta_{i_1}\cdots\theta_{i_k} \longmapsto f_{12\dots n} = \left(\text{coeff. of } \theta_1\cdots\theta_n\right) =: \int f$$

more generally: fix V - n-dim. v.space, an element $\mu \in \Lambda^n V$ "Berezinian"

then $\int^{\text{Berezin}} \mu \cdot - : \Lambda^\bullet V^* \longrightarrow \mathbb{R}$

$$f \longmapsto \langle \mu, f\rangle = \langle \mu, f^{(n)}\rangle$$

component in $\Lambda^n V^*$

pairing between $\Lambda^n V$ and $\Lambda^n V^*$

Ex: $(V=\mathbb{R})$ $\quad$ $V=\mathbb{R}^2$

$$\int^{\text{Berezin}} \underbrace{D\theta}_{\text{Berezinian}} (a+b\theta) = b, \quad \int^{\text{Berezin}} \underbrace{D\theta_2 D\theta_1}_{\text{Berezinian}} (\theta_1 + 5\theta_1\theta_2) = 5$$

- in V is an oriented real v.sp. with Euclidean metric, then there is a distinguished Berezinian

$\mu = e_n \wedge \dots \wedge e_1$

for $\{e_i\}$ any o/n, oriented basis in V.

- if V Eucl., oriented, $\Phi \in so(V)$ ~~$\simeq \Lambda^2 V$~~, then

i.e., $(x,\Phi y) = -(\Phi x, y)$

~~$\Phi \mapsto \sum_{i<j} (e_i, \Phi e_j)\, e_i \wedge e_j$~~

let $\hat\Phi = \frac{1}{2}\Phi_{ij}\theta_i\theta_j \in \Lambda^2 V^*$

θ_i: basis in V^* dual to an o/n basis $\{e_i\}$ in V

$$\int^{B} \underbrace{\mu}_{= D\theta_n\cdots D\theta_1} e^{\hat\Phi} = \underset{\text{Pfaffian}}{\mathrm{Pf}(\Phi)}$$

$$= \left\langle \mu, \frac{\hat\Phi^{[n/2]}}{[n/2]!} \right\rangle$$

Ex: $V=\mathbb{R}^2$ $\quad \Phi = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ $\quad \hat\Phi = a\,\theta_1\theta_2 \in \Lambda^2 V^*$

$$\int^{B} D\theta_2 D\theta_1\, e^{a\theta_1\theta_2} = a = \mathrm{Pf}(\Phi)$$

Let $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$ be an oriented rk$=n$ real v.bun. with fiber metric.

Let $\Xi \in \Gamma(E, \underbrace{\pi^* E}_{= T^{vert} E})$ be the tautological section,

$\Xi(x, \xi) = \xi$, $\xi \in E_x$

Fix a metric-compatible connection on E

with 1-form $\mathbb{A} \in \Omega^1(E, \pi^* E)$ and curvature $F \in \Omega^2(M, \underbrace{so(E)}_{\simeq \Lambda^2 E})$ — due to metric-compatibility

Let $\boxed{S = \underbrace{-\frac{1}{2}(\Xi, \Xi)}_{S_0} + \underbrace{\mathbb{A} + \pi^* F}_{S'}}$ $\in \bigoplus_{i=0}^{2} \Omega^i(E, \Lambda^i \pi^* E)$

$\subset Y = \bigoplus_{i,j} \underbrace{\Omega^i(E, \Lambda^j \pi^* E)}_{Y^{i,j}}$ — supercomm. algebra with product $=$ $\wedge$ on $\Omega^\bullet$ and $\wedge$ in coeffs

$\underset{Y^{i,j}}{\alpha} \wedge \underset{Y^{i',j'}}{\beta} \in Y^{i+i', j+j'}$

• We have a fiber Berezin integral map

$$\int^B : Y^{i,j} \longrightarrow \Omega^i(E)$$

$$\alpha \longmapsto \langle \pi^* \mu, \alpha \rangle$$ — vanishes unless $j = n$

with $\mu \in \Gamma(M, \Lambda^n E)$,
$\mu = e_n \wedge \dots \wedge e_1$ — o/n basis of sections of E

Set $\omega = C^n \underbrace{\int^B e^S}_{e^{S_0} \int^B e^{S'}} \in \Omega^\bullet(E)$, $C = \frac{1}{\sqrt{2\pi}}$ normalization factor

<u>Theorem</u> a) $\omega \in \Omega^n_{=}(E)$

b) $d\omega = 0$

c) $\int_{E_x} \omega = 1 \quad \forall x \in M$

i.e., ω is a "Gaussian-shaped" Thom form on E.

d) under a change of connection $\mathbb{A} \to \mathbb{A}'$, ω changes by $\omega \to \omega + d(\dots)$

Proof a) note: $S \in \bigoplus_i Y^{i,i} = Y^{diag}$ a comm. subalgebra in Y

$\Rightarrow e^S \in Y^{diag} \Rightarrow \Lambda^n \pi^* E$ -component of e^S (relevant for $\int^B$)

is also an n-form on E $\Rightarrow \int^B e^S \in \Omega^n(E)$ ✓

b) In a loc. trivialization of E, using an o/n basis of local sections $\{e_a\}$, we have

coords on E_x, i.e. $\xi = \xi_a e_a$

$$\boxed{S = -\frac{1}{2}\xi_a\xi_a + \theta_a(d\xi_a + A_{ab}\xi_b) + \frac{1}{2}\theta_a\theta_b F_{ab}}$$

generators of $\Lambda^{\bullet} E_x$ $\qquad$ $\in \Omega^1(M, so(n))$ loc. connection 1-form $\qquad$ $\in \Omega^2(M, so(n))$

consider the diff. operator $D := d_E + \xi_a \frac{\partial}{\partial \theta_a} - \theta_a A_{ab}\frac{\partial}{\partial \theta_b}$

One has: $DS = 0$ (short computation)

$\Rightarrow De^S = 0$ (*) (note: D is a derivation of Y)

- $\int^B D\alpha = d_E \int^B \alpha$ for any $\alpha \in Y$ (terms 2,3 in D when hitting the top-degree monomial in θ's produce lower-degree polynomials!)

here we are using $A_{ab} = -A_{ba}$ i.e. that $\mathcal{A}$ is metric-compatible

Thus: $\underbrace{d}_{d_E} \underbrace{\int^B e^S}_{\omega} = \int^B De^S = 0$ ✓

d) Let $\mathcal{A}_t$ be a path of connections on E, $t \in [0,1]$.

note that $\dot{\mathcal{A}}_t =: \alpha_t \in \Omega^1(M, so(E))$

One has $\frac{d}{dt} S_{\mathcal{A}_t} = \theta_a(\alpha_t)_{ab}\xi_b + \frac{1}{2}\theta_a\theta_b(d + [A_t, -])(\alpha_t)_{ab}$

$= D(\frac{1}{2}(\alpha_t)_{ab}\theta_a\theta_b)$ [short computation]

$\Rightarrow \frac{d}{dt}\int^B e^{S_{\mathcal{A}_t}} = \int^B \dot{S}_{\mathcal{A}_t} e^{S_{\mathcal{A}_t}} = \int^B D(\frac{1}{2}(\dot{\mathcal{A}}_t)_{ab}\theta_a\theta_b)e^{S_{\mathcal{A}_t}} \underset{\text{using (*)}}{=} \int^B D(\frac{1}{2}(\dot{\mathcal{A}}_t)_{ab}\theta_a\theta_b \cdot e^{S_{\mathcal{A}_t}})$

$= d_E\left(\int^B e^{S_{\mathcal{A}_t}} \frac{1}{2}\dot{\mathcal{A}}_t\theta\theta\right)$

$\Rightarrow \frac{d}{dt}\omega_{\mathcal{A}_t} = d_E(\dots) \underset{\int_0^1 dt}{\Rightarrow} \omega_{\mathcal{A}_1} - \omega_{\mathcal{A}_0} = d_E(\dots)$ ✓

c) $\int_{E_x} \omega = C^n \int_{E_x} \int^{B} e^{S} = C^n \int^{B} \int_{E_x} e^{-\frac{1}{2}\xi_a\xi_a} \left(\prod_a (1 + \underbrace{\theta_a d\xi_a}_{} + \underbrace{\theta_a A_{ab} d\xi_b}_{}) \right) e^{\frac{1}{2}\theta\theta F}$

no summation over a!

to get the top form on E_x, need to pick from each bracket

$= C^n \int^{B} \int_{E_x} e^{-\frac{1}{2}\xi_a\xi_a} \prod_a (\theta_a d\xi_a) \cdot \underbrace{e^{\frac{1}{2}\theta\theta F}}_{}$

can replace with 1, since we already have a top monomial in θ

$= C^n \underbrace{\int^{B} \theta_1 \cdots \theta_n}_{1} \underbrace{\int_{E_x} d\xi_1 \cdots d\xi_n \, e^{-\frac{1}{2}\xi_a\xi_a}}_{(\sqrt{2\pi})^n \text{ - Gaussian integral}} = 1 .$ ✓

$\boxtimes$

Mathai-Quillen
Euler form

If $E \xrightarrow{} M$ with s any section, then $\mathcal{E} = s^*\omega \in \Omega^n(M,\mathbb{R})$ represents the Euler class of E.

Explicitly:

$$\mathcal{E} = C^n s^* \int^{B} e^{S} = C^n \int^{B} e^{-\frac{1}{2}(s,s) + \nabla s + F}$$

where $-\frac{1}{2}(s,s) \in \Omega^0(M)$, $\nabla s \in \Omega^1(M,E)$, $F \in \Omega^2(M, \Lambda^2 E)$

$$= C^n \int^{B} e^{-\frac{1}{2} s_a s_a + \theta_a (d s_a + A_{ab} s_b) + \frac{1}{2}\theta_a\theta_b F_{ab}}$$

Ex: if $s = s_0$ the zero-section, then

$$s_0^* \omega = C^n \int^{B} e^{\frac{1}{2}\theta_a\theta_b F_{ab}} = \frac{1}{(2\pi)^{n/2}} \mathrm{Pf}(F) \in \Omega^n(M)$$

- Chern-Gauss-Bonnet representative for the Euler class.

With ε-parameter:

metric

$$S_\varepsilon = \frac{-1}{2\varepsilon} g(\Xi,\Xi) + \mathcal{A} + \varepsilon \pi^* g^{-1} \circ F$$

Locally: (using a non-o/n basis in E_x)

$$S_\varepsilon = \frac{-1}{2\varepsilon} g_{ab} \xi^a \xi^b + \theta_a (d\xi^a + A^a{}_b \xi^b) + \frac{\varepsilon}{2} F^a{}_c (g^{-1})^{cb} \theta_a \theta_b$$

$$\omega_\varepsilon = (2\pi\varepsilon)^{-\frac{n}{2}} \int^B \mu_g \, e^{S_\varepsilon} \quad , \qquad \mu_g = \sqrt{\det g}\, D\theta_n \cdots D\theta_1 \quad \in \Gamma(M, \Lambda^n E^*)$$

- Berezinian (independent of the choice of basis in E_x !)

- if $s: M \to E$ a section intersecting $\mathcal{E}_0$ transversally, then

$$s^* \omega_\varepsilon \xrightarrow[\varepsilon \to 0]{} \delta_{s^{-1}(0)} \in \Omega^\bullet_{\text{distrib.}}(M)$$

$\underline{Ex:}$ $E = M \times \mathbb{R}$, $A = 0$

$$S_\varepsilon = \frac{-1}{2\varepsilon} \xi^2 + \theta \, d\xi \quad \rightsquigarrow \quad \omega_\varepsilon = \frac{1}{\sqrt{2\pi\varepsilon}} \int D\theta \, \underbrace{e^{-\frac{1}{2\varepsilon}\xi^2 + \theta \, d\xi}}_{\theta \, d\xi \, e^{-\frac{1}{2\varepsilon}\xi^2}}$$

$$= \frac{1}{\sqrt{2\pi\varepsilon}} d\xi \, e^{-\frac{1}{2\varepsilon}\xi^2}$$

for $f: M \to \mathbb{R}$,

$$f^* \omega_\varepsilon = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{1}{2\varepsilon} f^2} df$$

Another version ("first-order formalism")

- for the Euler class

$$\begin{array}{c} E, g, \mathcal{A} \\ s \uparrow\downarrow \\ M \end{array}$$

$$s^* \omega_\varepsilon = \int_{E_x^*} \frac{d^n p}{\sqrt{g}} \int^{\text{Berezin}} \sqrt{g}\, D^n\theta \cdot e^{\langle p, s \rangle + \nabla s - \frac{\varepsilon}{2} g^{-1}(p,p) + \varepsilon g^{-1} \circ F}$$