

## LAST TIME

def The "tautological form" (or Liouville 1-form)  $\alpha \in \Omega^1(T^*X)$  is the 1-form given in cotangent coords by  $\alpha = \sum_i \xi_i dx_i$ . (\*)

The canonical symplectic form on  $T^*X$  is  $\omega := -d\alpha = \sum_i dx_i \wedge d\xi_i \in \Omega^2(T^*X)$

Coordinate-free definition:

$$M = T^*X \quad p = (x, \xi) \quad , \quad \xi \in T_x^*X$$

$$\begin{array}{ccc} \downarrow \pi & & \downarrow \\ X & & x \end{array}$$

- natural projection. Then:

$$\alpha_p := \underbrace{(d\pi_p)^*}_{\substack{T_p M \rightarrow T_x X \\ T_x^* X \rightarrow T_p^* M}} \xi \in T_p^* M \quad \text{Equivalently: } \alpha_p(v) := \xi \left( \underbrace{(d\pi_p(v))}_{\in T_x^* X} \right)$$

for any  $v \in T_p M$

in a cotangent chart,  $v = \sum_i v_i \left( \frac{\partial}{\partial x_i} \right)_p + w_j \left( \frac{\partial}{\partial \xi_j} \right)_p$

↑  
basis in  $T_p M$  induced by loc. coords  $x_i, \xi_j$

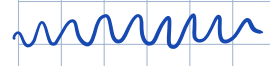
$$(d\pi)_p v = \sum_i v_i \left( \frac{\partial}{\partial x_i} \right)_x$$

$$\alpha_p(v) = \left( \sum_i \xi_i (dx_i)_x \right) \left( \sum_j v_j \left( \frac{\partial}{\partial \xi_j} \right)_x \right) = \sum_i \xi_i v_i = \left( \sum_i \xi_i (dx_i)_p \right) (v) = \alpha_p(v)$$

old def

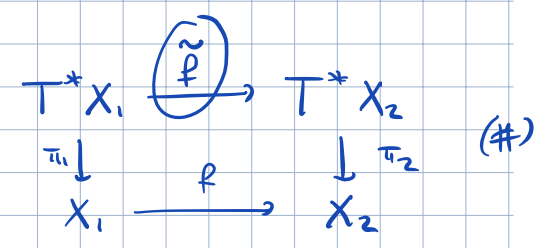
• canonical symplectic form  $\omega \in \Omega^2(T^*X)$

∴ defined as  $\omega = -d\alpha$ . Locally:  $\omega = \sum dx_i \wedge d\xi_i$



Naturality of taut. 1-form and can. symplectic str.

for a diffeo  $f: X_1 \rightarrow X_2$  one has a cotangent lift



in fibers:  $\tilde{f}: T_x^* X_1 \rightarrow T_{f(x)}^* X_2$

$$\left( (df)_x^* \right)^{-1} \quad (df)_x: T_x X_1 \rightarrow T_{f(x)} X_2$$

Lemma: a) If  $\alpha_{1,2}$  - taut. 1-forms on  $T^*X_{1,2}$ , then  $\alpha_1 = \tilde{f}^* \alpha_2$

b) Similarly, for canonical symplectic forms:  $\omega_1 = \tilde{f}^* \omega_2$  (i.e.,  $\tilde{f}$  is a symplectomorphism)

Proof: a) for  $p \in T^*X$ ,  $v \in T_p M$ ,

$$(\alpha_1)_p(v) = \sum \langle (d\pi_1)_p(v), \cdot \rangle$$

$$\begin{aligned}
 (\tilde{f}^* \alpha_2)_p(v) &= (\alpha_2)_{\tilde{f}(p)} \left( (d\tilde{f})_p v \right) \\
 &= \left( (d\tilde{f})_p^{-1} \right)^* \left( (d\pi_2)_{\tilde{f}(p)} (d\tilde{f})_p v \right) \\
 &= \sum \langle (d\pi_2)_{\tilde{f}(p)} (d\tilde{f})_p v, \cdot \rangle \\
 &= \sum \langle (d\pi_1)_p v, \cdot \rangle
 \end{aligned}$$

← using associativity of  $\langle \cdot, \cdot \rangle$

$$\Rightarrow \alpha_1 = \tilde{f}^* \alpha_2 \quad \checkmark$$

$$b) \omega_1 = -d\alpha_1 = -d\tilde{f}^* \alpha_2 = -\tilde{f}^* d\alpha_2 = \tilde{f}^* \omega_2 \quad \checkmark \quad \square$$

### Symplectic volume form

For  $(M, \omega)$  a  $2n$ -dim. symplectic manifold,

$$\nu := \frac{\omega^n}{n!} \in \Omega^{2n}(M) \quad \text{"symplectic volume form"} \quad , \quad \omega^n := \underbrace{\omega \wedge \dots \wedge \omega}_n$$

(or "Liouville volume form")

Locally, in a Darboux chart  $\omega = \sum_{i=1}^n dx_i \wedge dy_i$

$$\begin{aligned}
 \Rightarrow \nu &= \frac{\omega^n}{n!} = \frac{1}{n!} \sum_{\sigma \in S_n} (dx_{\sigma(1)} \wedge dy_{\sigma(1)}) \wedge \dots \wedge (dx_{\sigma(n)} \wedge dy_{\sigma(n)}) \\
 &= (dx_1 \wedge dy_1) \wedge \dots \wedge (dx_n \wedge dy_n) \quad \text{— a nonvanishing top-form.}
 \end{aligned}$$

• Thus,  $(M, \omega)$  has an orientation corresponding to  $\nu$ .

• For  $M$  compact,  $\text{vol}_{\text{symp}}(M, \omega) := \int_M \frac{\omega^n}{n!} > 0$  — symplectic volume

$$\Rightarrow [\omega^n] \in H^{2n}(M, \mathbb{R}) \text{ a nonzero class, moreover, } [\omega^n] = [\omega]^n,$$

thus  $[\omega] \in H^2(M, \mathbb{R})$  is a nonzero class.

- Corollary:
- sphere  $S^{2n}$  for  $n \geq 1$  doesn't admit a symplectic structure (compact but  $H^2$  vanishes, so  $[\omega]$  cannot be a nonzero class)
  - $\mathbb{R}P^2$  doesn't admit a symplectic structure (not orientable)

# Lagrangian submanifolds

a submanifold of  $M$  is a mfd  $X$  with a closed embedding  $i: X \hookrightarrow M$   
 = proper injective immersion  
 $i^{-1}(\text{compact set in } M) = \text{compact set in } X$

def Let  $(M, \omega)$  be a  $2n$ -dimensional symplectic mfd.

A submanifold  $Y$  of  $M$  is a Lagrangian submanifold if  $\forall p \in Y, T_p Y \subset T_p M$

is a Lagrangian subspace (i.e.  $\omega_p|_{T_p Y} = 0$  and  $\dim T_p Y = \frac{1}{2} \dim T_p M$ )

Equivalently,  $Y$  is Lagrangian iff  $i^* \omega = 0$  and  $\dim Y = \frac{1}{2} \dim M$ .  
 inclusion map  $i: Y \hookrightarrow M$

## Lagrangian submanifolds of $T^*X$

Let  $M = T^*X$  - cotangent bundle

Let  $X_0 = \{(x, \xi) \in T^*X \mid \xi = 0\}$  - zero-section  
 an  $n$ -dim. submanifold of  $X$

Let  $i_0: X_0 \hookrightarrow T^*X$  the inclusion. Then  $i_0^* \alpha \stackrel{\xi=0}{=} 0$   
 $\sum_i \xi_i dx_i$  is a cotangent form

also,  $i_0^* \omega = -d i_0^* \alpha = 0$

$\Rightarrow X_0$  is a Lagrangian submanifold of  $T^*X$

## Graph Lagrangians

Consider  $\mu \in \Omega^1(X)$ . Denote  $\check{\mu}: X \rightarrow T^*X$  the corresponding section of  $T^*X$   
 $x \mapsto \mu_x \in T_x^*X$

Lemma (#)  $\check{\mu}^* \alpha = \mu$   
 taut. 1-form on  $T^*X$

Proof  $(\check{\mu}^* \alpha)(v) = \alpha(d\check{\mu}_x(v)) \stackrel{\text{def of } \alpha}{=} \sum_i \xi_i ((d\pi)_p(d\check{\mu}_x(v))) = \mu_x(v)$   
 say  $\check{\mu}(x) = (x, \xi) = p$

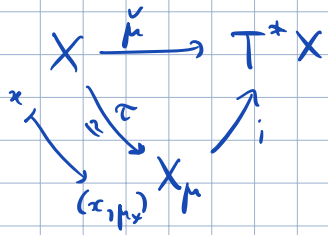
Let  $X_\mu := \{(x, \mu_x) \mid x \in X\} \subset T^*X$  (\*)

$\parallel$   
graph( $\mu$ ) - another notation

Proposition  $X_\mu \subset T^*X$  is a Lagrangian submanifold iff the 1-form  $\mu \in \Omega^1(M)$  satisfies  $d\mu = 0$

Proof: Denote  $i: X_\mu \hookrightarrow T^*X$  the inclusion.

$$i^*\omega = i^*(-d\alpha) = -d \underbrace{i^*\alpha}_{(\pi \circ i)^*} = -d(\pi^{-1})^* \underbrace{\mu}_\alpha \stackrel{\text{diff.}}{=} -(\pi^{-1})^* d\mu = -\mu \text{ by Lemma (\#)}$$



Thus:  $i^*\omega = 0$  iff  $d\mu = 0$ .

Rem In particular, one has Lagrangians  $X_\mu$  for  $\mu = df$ ,  $f \in C^\infty(X)$

In this case,  $f$  is called a generating function for the Lagrangian.

- If  $H^1(X) = 0$ , then all Lagrangians (\*) are of the form  $X_{df}$ .

• There are many Lagrangians in  $T^*X$  that are not of the form (\*), e.g.,  
cotangent fibers  $T_x^*X$ . For a fiber bundle  $\begin{matrix} E \\ \downarrow \pi \\ X \end{matrix}$ ,

Rem a submd  $i: Y \hookrightarrow E$  is said to be projectable if  $\pi \circ i: Y \rightarrow X$  is a diffeo.

Then: any projectable Lagrangian in  $T^*X$  is of the form (\*).

### Conormal bundles

Let  $S$  be any  $k$ -dimensional submanifold of  $X$ .

def The conormal space at  $x \in S$  is

$$N_x^*S = \{ \zeta \in T_x^*X \mid \zeta(v) = 0 \ \forall v \in T_x S \}$$

annihilator of a subspace  
 $= \text{Ann}(T_x S \subset T_x X)$

The conormal bundle of  $S$  is

$$N^*S = \{ (x, \zeta) \in T^*X \mid x \in S, \zeta \in N_x^*S \}$$

•  $N^*S$  is an  $n$ -dim. submanifold of  $T^*X$  (can prove using adapted loc. coordinates  
- coords  $x_1, \dots, x_n$  on  $X$  where  
 $S$  is given by  $x_{k+1} = \dots = x_n = 0$ )

Lemma Let  $i: N^*S \hookrightarrow T^*X$  be the inclusion.

Then  $i^*\alpha = 0$

Proof Let  $(U, x_1, \dots, x_n)$  be coords on  $X$  centered at  $x \in S$  and adapted so that <sup>to  $S$</sup>

$S \cap U$  is given by  $x_{k+1} = \dots = x_n = 0$ .

Let  $(T^*U, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$  - corresponding cotangent coord. system.

$N^*S \cap T^*U$  is given by  $x_{k+1} = \dots = x_n = 0, \xi_1 = \dots = \xi_k = 0$

Thus for  $p \in N^*S, \alpha_p|_{N^*S} = \sum_i \xi_i dx_i|_{N^*S} = \sum_{i>k} \xi_i dx_i|_{N^*S} = 0$   
 $\xi_1 = \dots = \xi_k = 0$       span  $\left\{ \frac{\partial}{\partial x_i} \right\}_{i < k}$        $\square$

Corollary: For any submanifold  $S \subset X$ , the conormal bundle  $N^*S$  is a Lagrangian submanifold of  $T^*X$

Ex: if  $S = \{x\}$  single point, then  $N^*\{x\} = T_x^*X$  the cotangent fiber

if  $S = X$ , then  $N^*X = X_0$  the zero-section.

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