LAST TIME

$(x_1, \ldots x_n, \xi_1, \ldots, \xi_n)$ on $T^*\mathcal{U}$

def The "tautological form" (or Liouville 1-form) $\alpha \in \Omega^1(T^*X)$ is the 1-form given in cotangent coords by $\alpha := \sum_i \xi_i \, dx_i$. (*)

The canonical symplectic form on T^*X is $\omega := -d\alpha = \sum_i dx_i \wedge d\xi_i \in \Omega^2(T^*X)$

Coordinate-free definition:

$$\begin{tikzcd} M = T^*X \arrow[d, "\pi"] & p=(x,\xi) \arrow[d] \\ X & x \end{tikzcd}$$

$, \xi \in T_x^*X$ - natural projection. Then:

$$\boxed{\alpha_p := \underbrace{\underbrace{(d\pi_p)}_{T_pM \to T_xX}{}^*}_{T_x^*X \to T_p^*M} \overset{\in T_x^* X}{\xi}} \in T_p^*M .$$

Equivalently: $\boxed{\alpha_p(v) := \xi\Big(\underbrace{(d\pi)_p(v)}_{\in T_xM}\Big)}$ for any $v \in T_pM$

- in a cotangent chart, $v = \sum_i v_i \underbrace{\left(\frac{\partial}{\partial x_i}\right)_p + w_i \left(\frac{\partial}{\partial \xi_i}\right)_p}_{\text{basis in } T_pM \text{ induced by loc. coords } x_i, \xi_i}$

$$(d\pi)_p v = \sum_i v_i \left(\frac{\partial}{\partial x_i}\right)_x$$

$\Downarrow$

$$\alpha_p(v) \overset{\text{new def}}{=} \left(\sum_i \xi_i (dx_i)_x\right)\left(\sum_j v_j \left(\frac{\partial}{\partial \xi_j}\right)_x\right) = \sum_i \xi_i v_i = \left(\sum_i \xi_i (dx_i)_p\right)(v) \overset{\text{old def}}{=} \alpha_p(v).$$

- canonical symp. form $\omega \in \Omega^2(T^*X)$ is defined as $\omega = -d\alpha$. Locally: $\omega = \sum_i dx_i \wedge d\xi_i$

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Naturality of taut. 1-form and can. symp. str.

for a diffeo $f : X_1 \xrightarrow{\sim} X_2$ one has a cotangent lift

$$\begin{tikzcd} T^*X_1 \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"] & T^*X_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd} \quad (\#)$$

in fibers: $\tilde{f} : T_x^*X_1 \to T_{f(x)}^*X_2$

$\qquad \| $

$\quad ((df)_x^*)^{-1} \qquad (df)_x : T_xX_1 \to T_{f(x)}X_2$

Lemma: a) If $\alpha_{1,2}$ - taut. 1-forms on $T^*X_{1,2}$, then $\boxed{\alpha_1 = \tilde{f}^*\alpha_2}$.

b) Similarly, for canonical symp. forms: $\omega_1 = \tilde{f}^*\omega_2$ (i.e., $\tilde{f}$ is a symplectomorphism)

Proof: a) for $\overset{(x,\xi)}{p} \in \underset{M_1}{T^*X_1}$, $v \in T_pM_1$,

$(\alpha_1)_p(v) = \xi((d\pi_1)_p(v))$

$$(\tilde{f}^*\alpha_2)_p(v) = (\alpha_2)_{\overset{\substack{f(x),\,((df)_x^*)^{-1}\xi \\ \downarrow}}{\tilde{f}(p)}}((d\tilde{f})_p v)$$

$$= \left(\left(\cancel{(df)_x^{-1}}\right)^*\xi\right)\left(\underbrace{(d\pi_2)_{\tilde{f}(p)}(d\tilde{f})_p v}_{\cancel{(df)}_x (d\pi_1)_p\, v}\right)$$ ← using commutativity of (*)

$$= \xi(d\pi_1)_p v)$$

$\Rightarrow \alpha_1 = \tilde{f}^*\alpha_2$ ✓

b) $\omega_1 = -d\alpha_1 = -d\tilde{f}^*\alpha_2 = -\tilde{f}^*d\alpha_2 = \tilde{f}^*\omega_2$ ✓ □

Symplectic volume form

For (M,ω) a $2n$-dim. symp. mfd,

$\nu := \frac{\omega^n}{n!} \in \Omega^{2n}(M)$ – "symplectic volume form" (or "Liouville volume form"), $\omega^n := \underbrace{\omega\wedge\dots\wedge\omega}_{n}$

Locally, in a Darboux chart $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$

$$\Rightarrow \nu = \frac{\omega^n}{n!} = \frac{1}{n!}\sum_{\sigma\in S_n}(dx_{\sigma(1)}\wedge dy_{\sigma(1)})\wedge\dots\wedge(dx_{\sigma(n)}\wedge dy_{\sigma(n)})$$

$$= (dx_1\wedge dy_1)\wedge\dots\wedge(dx_n\wedge dy_n)$$ – a nonvanishing top-form.

- Thus, (M,ω) has an orientation corresponding to ν.
- for M compact, $\mathrm{vol}_{\mathrm{symp}}(M,\omega) := \int_M \frac{\omega^n}{n!} > 0$ – symplectic volume

  $\Rightarrow [\omega^n] \in H^{2n}(M,\mathbb{R})$ a nonzero class, moreover, $[\omega^n] = [\omega]^n$,

  thus $[\omega] \in H^2(M,\mathbb{R})$ is a nonzero class.

Corollary:
- sphere S^{2n} for $n>1$ doesn't admit a symplectic structure (compact but H^2 vanishes, so $[\omega]$ cannot be a nonzero class)
- $\mathbb{RP}^2$ doesn't admit a symp. structure (not orientable)

Lagrangian submanifolds

- a submanifold of M is a mfd X with a $\underbrace{\text{closed}}\ \underline{\text{embedding}}$ $i: X \hookrightarrow M$

  $= \underbrace{\text{proper}}$ injective immersion

  i^{-1}(compact set in M) $=$ compact set in X

def Let (M,ω) be a $2n$-dimensional symp. mfd.
A submanifold Y of M is a Lagrangian submanifold if $\forall p \in Y$, $T_pY \subset T_pM$ is a Lagrangian subspace (i.e. $\omega_p|_{T_pY} = 0$ and $\dim T_pY = \frac{1}{2}\dim T_pM$)

Equivalently, Y is Lagrangian iff $\underset{\uparrow}{i^*}\omega = 0$ and $\dim Y = \frac{1}{2}\dim M$.

inclusion map $i: Y \hookrightarrow M$

Lagrangian submanifolds of T^*X

Let $M = T^*X$ - cotangent bundle

Let $X_0 = \{(x, \underset{\overset{\cap}{T^*_xX}}{\xi}) \in T^*X \mid \xi = 0\}$ - zero-section
- an n-dim. submanifold of X

Let $i_0: X_0 \hookrightarrow T^*X$ the inclusion. Then $i_0^*\underbrace{\alpha}_{\sum_i \xi_i dx_i} \overset{\text{set } \xi := 0}{=} 0$

in a cotangent chart

also, $i_0^*\underbrace{\omega}_{-d\alpha} = -d\, i_0^*\alpha = 0$

$\Rightarrow$ $\boxed{X_0 \text{ is a Lagrangian submanifold of } T^*X}$

Graph Lagrangians

- Consider $\mu \in \Omega^1(X)$. Denote $\check{\mu}: X \to T^*X$, $x \mapsto \mu_x \in T^*_xX$ the corresponding section of

$$\begin{tikzcd} T^*X \arrow[d] \\ X \arrow[u, bend left, dashed, "\check{\mu}"] \end{tikzcd}$$

Lemma (#) $\boxed{\check{\mu}^* \overset{\overset{\Omega^1(T^*X)}{\Downarrow}}{\alpha} = \overset{\overset{\Omega^1(X)}{\Downarrow}}{\mu}}$

taut. 1-form on T^*X

Proof $(\check{\mu}^*\alpha)(\underset{\overset{\cap}{T_xX}}{v}) = \alpha(d\check{\mu}_x(v)) \overset{\text{def of } \alpha}{=} \xi(\underbrace{(d\pi)_p(d\check{\mu})_x}_{id}(v)) = \mu_x(v)$

say $\check{\mu}(x) = (x, \underset{\overset{\shortparallel}{\mu_x}}{\xi}) = p$

Let $\underset{\substack{\| \\ \text{graph}(\mu)}}{X_\mu} := \{(x, \underset{\substack{\cap \\ T^*_x X}}{\mu_x}) \mid x \in X\} \subset T^*X$ (*)

graph(μ) - another notation

<u>Proposition</u> $X_\mu \subset T^*X$ is a Lagrangian submanifold iff the 1-form $\mu \in \Omega^1(M)$ satisfies $\boxed{d\mu = 0}$

⌈ <u>Proof</u>: Denote $i : X_\mu \hookrightarrow T^*X$ the inclusion.

$$i^*\omega = i^*(-d\alpha) = -d \underbrace{i}_{(\check{\mu}\circ\tau^{-1})}{}^* \alpha = -d\,(\tau^{-1})^* \underbrace{\check{\mu}^*\alpha}_{=\mu \text{ by Lemma } (\#)} = -(\overset{\substack{\text{diffeo} \\ \downarrow}}{\tau^{-1}})^* d\mu$$

$$\begin{tikzcd} X \arrow[rr, "\check{\mu}"] \arrow[dr, "\tau", "\cong"'] & & T^*X \\ & X_\mu \arrow[ur, "i"'] & \end{tikzcd}$$

$x \mapsto (x, \mu_x)$

Thus: $i^*\omega = 0$ iff $d\mu = 0$.

□ ⌋

<u>Rem</u> In particular, one has Lagrangians X_μ for $\mu = df$, $f \in C^\infty(X)$

In this case, f is called a <u>generating function</u> for the Lagrangian.

- If $H^1(X) = 0$, then all Lagrangians (*) are of the form X_{df}.

- There are many Lagrangians in T^*X that are <u>not</u> of the form (*), e.g., cotangent fibers $T^*_x X$.

<u>Rem</u> For a fiber bundle $\begin{smallmatrix} E \\ \downarrow \pi \\ X \end{smallmatrix}$, a submfd $i : Y \hookrightarrow E$ is said to be <u>projectable</u> if $\pi \circ i : Y \to X$ is a diffeo.
Then: any <u>projectable</u> <u>Lagrangian</u> in T^*X is of the form (*).

<u>Conormal bundles</u>

Let S be any k-dimensional submanifold of X.

<u>def</u> The conormal space at $x \in S$ is

$$N^*_x S = \{\xi \in T^*_x X \mid \xi(v) = 0 \ \forall v \in T_x S\} = \overset{\substack{\text{annihilator of a subspace} \\ \swarrow}}{\text{Ann}}(T_x S \subset T_x X)$$

The conormal bundle of S is

$$N^*S = \{(x, \xi) \in T^*X \mid x \in S,\ \xi \in N^*_x S\}$$

- N^*S is an n-dim. submanifold of T^*X (can prove using adapted loc. coordinates – coords $x_1 \dots x_n$ on X where S is given by $x_{k+1} = \dots = x_n = 0$)

Lemma Let $i : N^*S \hookrightarrow T^*X$ be the inclusion.

Then $i^*\alpha = 0$

Proof Let $(\mathcal{U}, x_1, \dots, x_n)$ be coords on X centered at $x \in S$ and adapted to S so that $S \cap \mathcal{U}$ is given by $x_{k+1} = \dots = x_n = 0$.

Let $(T^*\mathcal{U}, x_1 \dots x_n, \xi_1 \dots \xi_n)$ - corresponding cotangent coord. system.

$N^*S \cap T^*\mathcal{U}$ is given by $x_{k+1} = \dots = x_n = 0$, $\xi_1 = \dots = \xi_k = 0$

Thus for $p \in N^*S$,
$$\alpha_p \Big|_{N^*S} = \sum_i \xi_i \, dx_i \Big|_{N^*S} \underset{\substack{\uparrow \\ \xi_1 = \dots = \xi_k = 0}}{=} \sum_{i>k} \xi_i \, dx_i \Big|_{\mathrm{Span}\left\{\frac{\partial}{\partial x_i}\right\}_{i<k}} = 0$$
□

Corollary: For any submanifold $S \subset X$, the conormal bundle N^*S is a Lagrangian submanifold of T^*X

Ex: . if $S = \{x\}$ single point, then $N^*\{x\} = T^*_x X$ the cotangent fiber

. if $S = X$, then $N^*X = X_0$ the zero-section.