"Graph of a symplectomorphism is Lagrangian"

Let (M_1, ω_1), (M_2, ω_2) be two $2n$-dimensional symp. mflds., let $\varphi: M_1 \to M_2$ be a diffeomorphism.

Let $\Gamma_\varphi := \mathrm{graph}(\varphi) = \{(p, \varphi(p)) \mid p \in M_1\} \subset M_1 \times M_2$

Γ_φ = image of the embedding $i: M_1 \longrightarrow M_1 \times M_2$
$p \longmapsto (p, \varphi(p))$

$$\begin{array}{ccc} & M_1 \times M_2 & \\ {\scriptstyle pr_1} \swarrow & & \searrow {\scriptstyle pr_2} \\ M_1 & & M_2 \end{array}$$

Let $\tilde{\omega} := -pr_1^*(\omega_1) + pr_2^*(\omega_2)$ - "twisted" product symplectic form on $M_1 \times M_2$

<u>Proposition</u> $\varphi: M_1 \to M_2$ is a symplectomorphism iff Γ_φ is a Lagrangian submanifold of $(M_1 \times M_2, \tilde{\omega})$

<u>Proof</u> $i^* \tilde{\omega} = -\underbrace{i^* pr_1^*}_{(pr_1 \circ i)^* = id} \omega_1 + \underbrace{i^* pr_2^*}_{(pr_2 \circ i)^* = \varphi^*} \omega_2 = -\omega_1 + \varphi^* \omega_2 = 0$ iff φ is a symplectomorphism $\square$

(and $\dim \Gamma_\varphi = \frac{1}{2} \dim(M_1 \times M_2)$)

<u>Rem</u> more generally, one calls a Lagrangian $L \subset \overline{M_1} \times M_2$ (reversal of symp str.; $-pr_1^*(\omega_1)+pr_2^*(\omega_2)$)

a "canonical relation" between (M_1,ω_1) and (M_2,ω_2). Notation: $M_1 \xrightarrow{L} M_2$.

(or "Lagrangian correspondence")

Canonical rel that is projectable to M_2 (by pr_2) = graph of a symplectomorphism $\varphi: M_1 \to M_2$

Generating function for a symplectomorphism

Let X_1, X_2 two n-manifolds. Fix $f \in C^\infty(X_1 \times X_2)$

$$\mathrm{graph}(df) \subset \underbrace{T^*X_1 \times T^*X_2}_{pr_1^*(\omega_1)+pr_2^*(\omega_2)} \quad \text{- Lagrangian}$$

(canonical symp. forms)

Let $\sigma_1: T^*X_1 \to T^*X_1$ - symplectomorphism $(T^*X_1,\omega_1) \to (T^*X_1,-\omega_1)$

$(x,\xi) \mapsto (x,-\xi)$

Let $\sigma = \sigma_1 \times id: T^*X_1 \times T^*X_2 \to \overline{T^*X_1} \times T^*X_2$

Then: $L_f := \sigma \circ \mathrm{graph}(df) \subset \overline{T^*X_1} \times T^*X_2$ Lagrangian

$-pr_1^*(\omega_1) + pr_2^*(\omega_2)$

in a loc. chart on X_1, X_2:

$$L_f = \left\{ \left(x_i, \xi_i = -\frac{\partial f(x,y)}{\partial x_i}, y_i, \eta_i = \frac{\partial f(x,y)}{\partial y_i}\right) \right\}$$

$i=1..n$

If L_f is a graph of a diffeo $\varphi: T^*X_1 \to T^*X_2$, then φ is a symplectomorphism

f is the "generating function" for φ.

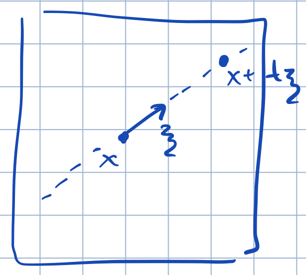
(Local necessary condition for f to generate a diffeo: $\det\left(\frac{\partial^2 f}{\partial x_i \partial y_j}\right) \neq 0$ - necessary for implicit fun. thm, to solve $\xi_i = -\frac{\partial f(x,y)}{\partial x_i}$ for y)

<u>Ex:</u> $X_1 = X_2 = \mathbb{R}^n$, $f(x,y) = \frac{\|x-y\|^2}{2t}$

solve for y in terms of x, ξ : $y = x + t\xi$

$\Rightarrow L_f = \{(x_i, \xi_i = -\frac{1}{t}(x_i - y_i), y_i, \eta_i = \frac{1}{t}(y_i - x_i))\} = \{(x_i, \xi_i, x_i + t\xi_i, \eta_i = \xi_i)\}$

$$\Rightarrow \varphi : T^*X_1 \longrightarrow T^*X_2$$
$$(x, \xi) \longrightarrow (x + t\xi, \xi)$$



Moser's trick & Darboux theorem

def For M a mfd, an isotopy is a smooth map $\rho : \underset{[0,1]}{I} \times M \to M$ such that a) the family of maps $\rho_t = \rho(t,-) : M \to M$ are diffeomorphisms;

b) $\rho_0 = id_M$

Given an isotopy ρ_t, one has a time-dependent vector field X_t on M,

$$\boxed{X_t(p) = \frac{d}{ds}\Big|_{s=t} \rho_s(\rho_t^{-1}(p))} \quad (*)$$

ρ_t = "flow" of the time-dependent v field X_t (though doesn't satisfy $\rho_s \circ \rho_t = \rho_{s+t}$):

setting $\gamma(t) = \rho_t(p)$, we have $\boxed{\dot\gamma(t) = X_t(\gamma(t))}$

- given a compactly supported X_t, one can contuct an isotopy s.t. $(*)$ holds

? - if X_t is not c.s., isotopy exists locally, near each p.

Lemma Let $\rho_t : M \to M$ be an isotopy and α_t a smooth family of k-forms.

Then:
$$\frac{d}{dt}\rho_t^* \alpha_t = \rho_t^*\left(\mathcal{L}_{X_t}\alpha_t + \frac{d\alpha_t}{dt}\right)$$

Proof: for $\alpha \in \Omega^k(M)$ a time-indep. form,

we have: $\frac{d}{dt}\rho_t^*\alpha = \rho_t^* \mathcal{L}_{X_t}\alpha$

$$\Rightarrow \frac{d}{dt}\rho_t^*\alpha_t = \frac{d}{dx}\Big|_{x=t}\rho_x^*\alpha_t + \frac{d}{dy}\Big|_{y=t}\rho_t^*\alpha_y = \rho_t^*\mathcal{L}_{X_t}\alpha_t + \rho_t^*\left(\frac{d\alpha_t}{dt}\right)$$

chain rule $\Rightarrow$ for $f(x,y)$ a fun of two vars,

$$\frac{d}{dt}f(t,t) = \frac{d}{dx}\Big|_{x=t}f(x,t) + \frac{d}{dy}\Big|_{y=t}f(t,y)$$

$\square$

<u>Moser's trick</u> Suppose $\alpha_0, \alpha_1 \in \Omega^k(M)$ two given k-forms and we are trying to find a diffeo $\varphi: M \to M$ s.t. $\boxed{\varphi^* \alpha_1 = \alpha_0}$ (##)

<u>Idea:</u> Let α_t be a family of k-forms connecting α_0 and α_1.

Construct an isotopy $\varphi_t : M \to M$ s.t. $\boxed{\varphi_t^* \alpha_t = \alpha_0}$ (#)

(#) $\Leftrightarrow 0 = \frac{d}{dt}(\varphi_t^* \alpha_t) = \varphi_t^*(\mathcal{L}_{X_t} \alpha_t + \dot{\alpha}_t)$ so, we need to find a family X_t s.t.

(@) $\mathcal{L}_{X_t} \alpha_t + \dot{\alpha}_t = 0$ then flow of X_t from $t=0$ to $t=1$ is the isotopy φ intertwining α_0 and α_1 (##).

<u>example:</u>

<u>Theorem</u> (Moser)

Let M be an oriented compact smooth manifold and α_0, α_1 two volume forms on M. (compatible with orientation)

Then there exists a (orientation preserving) diffeo $\varphi: M \to M$ s.t. $\varphi^* \alpha_1 = \alpha_0$ iff $\int_M \alpha_0 = \int_M \alpha_1$.

<u>Proof</u> ($\Rightarrow$) assume φ exists. then $\int_M \alpha_0 = \int_M \varphi^*(\alpha_1) = \int_{\varphi(M)} \alpha_1 = \int_M \alpha_1$. ✓

($\Leftarrow$) assume $\int_M \alpha_0 = \int_M \alpha_1 \Rightarrow \int_M (\alpha_1 - \alpha_0) = 0 \Rightarrow [\alpha_1 - \alpha_0] = 0 \in H^n_{de\,Rham}(M)$

$\Rightarrow \exists \beta \in \Omega^{n-1}(M)$ s.t. $\alpha_1 - \alpha_0 = d\beta$. Set $\alpha_t = (1-t)\alpha_0 + t\alpha_1$.

Each α_t is a volume form: $(\alpha_t)_p(\underbrace{v_1, \dots, v_n}_{\text{oriented basis at } p}) = (1-t)\underbrace{\alpha_0(v_1, \dots, v_n)}_{>0} + t\,\alpha_1(v_1, \dots, v_n) > 0$

$\dot{\alpha}_t = \alpha_1 - \alpha_0 = d\beta$

Moser's equation: $0 = \mathcal{L}_{X_t} \alpha_t + \dot{\alpha}_t = d\,\iota_{X_t} \alpha_t + d\beta$

- we can find X_t satisfying $\iota_{X_t} \alpha_t = -d\beta$ since α_t are vol. forms

$\xRightarrow{\text{Moser's trick}} \varphi = \mathrm{Flow}_{t=0}^{t=1}\{X_t\} : M \xrightarrow{\sim} M$ intertwines α_0 and α_1 ☒

<u>Corollary</u> Let $(M_1, \omega_1), (M_2, \omega_2)$ be two closed 2-dimensional symplectic manifolds.

Then they are symplectomorphic iff they have same genus and symplectic area.

<u>Proof</u> ($\Rightarrow$) $\exists \varphi: M_1 \xrightarrow{\sim} M_2$ symplectomorphism $\Rightarrow \varphi$ is a diffeo $\Rightarrow$ genus is the same

$\int_{M_1} \omega_1 = \int_{M_1} \varphi^* \omega_2 = \int_{M_2} \omega_2$

($\Leftarrow$) if M_1, M_2 have same genus, there is an or-preserving diffeo $\psi : M_1 \xrightarrow{\sim} M_2$

$\rightsquigarrow$ we have two symp forms $\omega_1, \psi^*\omega_2$ on M_1 with same symp. area

$\Rightarrow$ (Moser thm) $\exists\ F : M_1 \xrightarrow{\sim} M_1$ (diffeo) s.t. $\omega_1 = F^*(\psi^*\omega_2) = (\psi \circ F)^*\omega_2$ $\Rightarrow$ $\psi \circ F : M_1 \to M_2$ is the symplectomorphism $\square$

Another application of Moser trick:
changing the symp. form within the same class in H^2_{dR} yields a symplectomorphic manifold

<u>Thm</u> Let M be compact and $\omega_t = \omega_0 + d\beta_t$ (with $\beta_0 = 0$) a smooth family of symp. forms on M.
Then there exists an isotopy $\varphi_t : M \to M$ s.t. $\varphi_t^*\omega_t = \omega_0$

<u>Proof</u> Use Moser's trick. $(@) : 0 = \mathcal{L}_{X_t}\omega_t + \dot{\omega}_t = d(\iota_{X_t}\omega_t + \dot{\beta}_t)$

$\Leftarrow$ $\iota_{X_t}\omega_t = -\dot{\beta}_t$ can solve for X_t (by non-degeneracy of ω_t)

$\Rightarrow$ $\varphi_t = \mathrm{Flow}\{X_t\}$ is the isotopy intertwining ω_0 and ω_t. $\square$