ERRATUM to LAST TIME:

The "example" $(G = S^1 \times S^1) \circlearrowright (\underset{q\ ,\ p}{S^1 \times S^1}, \omega = dq \wedge dp) \xrightarrow{\mu} \mathfrak{g}^*$ of a non-equivariant moment map

$$(q,p) \longmapsto (q,p)$$

does not work, since q, p are not globally defined (single-valued) functions on M.

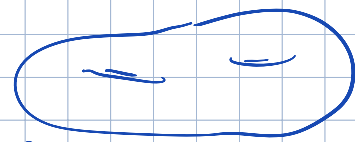
- in fact, this G-action is only symplectic, not Hamiltonian.

(can easily convert it to a valid example with $(G = \mathbb{R}^2) \circlearrowright (M = T^*\mathbb{R}) \xrightarrow{\mu} \mathfrak{g}^*$)

$$(q,p) \longmapsto (q,p)$$

with non-equivar. moment map

LAST TIME



Example (Atiyah-Bott) Let Σ - closed oriented surface

G a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ with ad-invariant $\overset{(\text{non-deg.})}{\text{inner}}$ product $(,)$.

Then: $\mathrm{Conn} = \left\{ \text{connections } A \text{ in } \begin{array}{c} \Sigma \times G \\ \downarrow \text{triv.} \\ \Sigma \end{array} \right\}$ form a $\overset{(\infty\text{-dim.})}{\text{symplectic}}$ manifold,

$T_A \mathrm{Conn} \simeq \Omega^1(M, \mathfrak{g})$

$\omega_A(\underbrace{\alpha, \beta}_{\in\, \Omega^1(M,\mathfrak{g})}) := \int_\Sigma (\alpha \overset{\wedge}{,} \beta)$

$\underset{= \mathcal{G}}{\mathrm{Map}(\Sigma, G)} \circlearrowright \mathrm{Conn}$ - action of bundle automorphisms on connections

$\Big(\underset{\underset{\sigma_0^* A}{\uparrow}}{A} \to {}^g A = g A g^{-1} + g\, d g^{-1}$ ← for a matrix group G

$= \mathrm{Ad}_g A + g^* \theta_{\text{Maurer-Cartan}} \Big)$

- This action is Hamiltonian,
 with moment map $\mu : \mathrm{Conn} \xrightarrow{\text{curvature}} \Omega^2(M, \mathfrak{g})$

$A \longmapsto F_A = dA + \frac{1}{2}[A, A]$

Idea: $\psi(\xi)\big|_{A \in \mathrm{Conn}} = -d\xi - [A, \xi] \in T_A \mathrm{Conn} \simeq \Omega^1(M, \mathfrak{g})$ - vector field of infin. action, evaluated at a point $A \in \mathrm{Conn}$

$\xi \in \mathrm{Map}(\Sigma, \mathfrak{g}) = \mathrm{Lie}(\mathcal{G})$

$\underset{\text{any}}{\iota_\rho}\, \iota_{\psi(\xi)} \omega \big|_A = -\int_\Sigma (d\xi + [A, \xi], \underset{\Omega^1(M,\mathfrak{g}) \simeq T_A \mathrm{Conn}}{\rho})$ ✓

$\iota_\rho \delta \langle \mu, \xi \rangle = \int_\Sigma \iota_\rho \delta (\xi, dA + \frac{1}{2}[A, A]) = \int_\Sigma (\xi, d\rho + [\rho, A]) \underset{\text{Stokes'}}{=} -\int_\Sigma (d\xi + [A, \xi], \rho)$

$\Rightarrow \langle \mu, \xi \rangle$ is a Hamiltonian for $\psi(\xi)$

• μ is equivariant (by transformation property of curvature under gauge transformations)

Rem: For $\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ \Sigma \end{array}$ a nontrivial bundle, CdS, s.25

$\underset{= \mathcal{G}}{\mathrm{Aut}(P)} \circlearrowright \mathrm{Conn}(P), \omega_{AB} \xrightarrow{\mu = \text{curvature}} \Omega^2(M, \mathrm{ad}(P))$

$\mathrm{Lie}(\mathcal{G}) \simeq \Gamma(M, \mathrm{ad}(P))$

Existence & uniqueness of moment maps

- Lie algebra cohomology: (Chevalley-Eilenberg cohomology) for $\mathfrak{g}$ a Lie algebra, A a module, $C^k_{CE}(\mathfrak{g}, A) := \Lambda^k \mathfrak{g}^* \otimes A = \{\varphi : \Lambda^k \mathfrak{g} \to A\}$ – Chevalley-Eilenberg cochains

differential: for $\varphi \in C^k$, $(d_{CE}\varphi)(x_0, \dots, x_k) = \sum_{0 \le i < j \le k} (-1)^{i+j} \varphi([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots x_k)$

$+ \sum_i (-1)^i \underbrace{x_i \cdot}_{\text{module action}} \varphi(x_0 \dots \hat{x}_i \dots x_k)$

e.g. for $A = \mathbb{R}$ (triv. module),

$C^\bullet_{CE} : \left(\overset{0}{\mathbb{R}} \xrightarrow{0} \overset{1}{\mathfrak{g}^*} \xrightarrow[-[,]^*]{} \overset{2}{\Lambda^2 \mathfrak{g}^*} \to \Lambda^3 \mathfrak{g}^* \to \dots \to \Lambda^{\dim \mathfrak{g}} \mathfrak{g}^* \right) = \Lambda^\bullet \mathfrak{g}^*$, d_{CE} = extension of $\mathfrak{g}^* \xrightarrow{-[,]^*} \Lambda^2 \mathfrak{g}^*$ to a derivation

- a Lie algebra $\mathfrak{g}$ is called "simple" if it does not have ideals I (apart from 0 and $\mathfrak{g}$ itself)
- —"— "semi-simple" if $\mathfrak{g} = \bigoplus_k \mathfrak{g}_k$ with $\mathfrak{g}_k$ simple.

__Whitehead lemma__: For $\mathfrak{g}$ semi-simple:

(1) $H^1_{CE}(\mathfrak{g}, \mathbb{R}) = 0$ $\iff$ $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ (:= ideal generated by elements of form $[x,y]$, $x, y \in \mathfrak{g}$))

(2) $H^2_{CE}(\mathfrak{g}, \mathbb{R}) = 0$

__Thm__ Let $\Phi : G \times M \to M$ be an action of G on M by symplectomorphisms and let $H^1(\mathfrak{g}, \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = 0$. Then: action Φ is Hamiltonian and has a unique equivar. moment map

__Proof__ (a) $H^1(\mathfrak{g}, \mathbb{R}) = 0 \iff \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \Rightarrow \underset{\in \mathfrak{g}}{\psi(X)} = \psi\left(\sum_i [Y_i, Z_i]\right) = \sum_i [\underset{\text{symp. v.f.}}{\psi(Y_i)}, \psi(Z_i)] \in \mathfrak{X}_{Ham}(M)$ recall: a commutator of symp. v.f. is Hamiltonian!

$\Rightarrow \exists\, \underset{\text{possibly non-equivariant}}{\mu^*} : \mathfrak{g} \to C^\infty(M)$ s.t. $\iota_{\psi(X)} \omega = d\mu^*(X)$

(b) Let $c(X,Y) = \underbrace{\mu^*([X,Y])}_{\text{Ham. for } [\psi(X), \psi(Y)]} - \{\mu^*(X), \mu^*(Y)\}$ = Ham. for zero v.field $\in \mathbb{R}$ (assume M is connected)
$X, Y \in \mathfrak{g}$

$c \in C^2_{CE}(\mathfrak{g})$; Jacobi identity in $\mathfrak{g}$ $\Rightarrow$ $d_{CE} c = 0$ $\underset{H^2_{CE}(\mathfrak{g}) = 0}{\Longrightarrow}$ $c = d_{CE}(b)$, $b \in C^1_{CE}(\mathfrak{g}) = \mathfrak{g}^*$

So: $c(X,Y) = b([X,Y])$

set $\tilde{\mu}^*(X) := \mu^*(X) + b(X)$ – satisfies the Lie alg. homom. property $\tilde{\mu}^*([X,Y]) - \{\tilde{\mu}^*(X), \tilde{\mu}^*(Y)\} = 0$

$\Rightarrow \tilde{\mu}^*$ is equivariant (co-) moment map.

(c) Let $\mu_{1,2}$ two equiv. moment maps

$(\mu_2^* - \mu_1^*) : \mathfrak{g} \to \underset{\text{Ham. for zero v.field}}{\mathbb{R}} \subset C^\infty(M)$ (const. fun.) $\Rightarrow \mu_2^* - \mu_1^* = a \in \mathfrak{g}^*$

$\mu_2^*([X,Y]) \overset{\text{equiver. of } \mu_2}{=} \{\mu_2^*(X), \mu_2^*(Y)\}$

$\Rightarrow \mu_1^* = \mu_2^*$ on $[\mathfrak{g}, \mathfrak{g}]$ $\underset{H^1(\mathfrak{g}) = 0}{\Rightarrow}$ $\mu_1^* = \mu_2^*$ ∎

$\mu_1^*([X,Y]) \underset{\text{eq. of } \mu_1}{=} \{\mu_1^*(X), \mu_1^*(Y)\}$

Symplectic reduction

Theorem (Marsden–Weinstein–Meyer)
Let (M,ω,G,μ) be a Hamiltonian G-space with G a compact Lie group and μ an equivariant moment map. Let $i:\mu^{-1}(0)\hookrightarrow M$ be the inclusion map.
Assume that G acts freely on $\mu^{-1}(0)$. Then

(1) · The orbit space $M_{red}=\mu^{-1}(0)/G$ is a manifold

(2) · $\pi:\mu^{-1}(0)\to M_{red}$ is a principal G-bundle

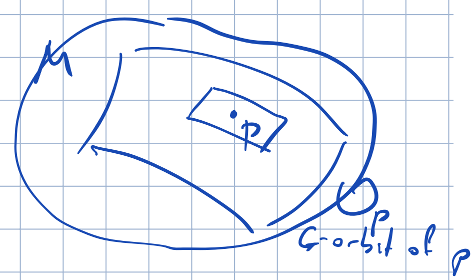
(3) · there is a symplectic form ω_{red} on M_{red} s.t. $\pi^*\omega_{red}=i^*\omega$.

def (M_{red},ω_{red}) is called the reduction (or "symplectic quotient") of (M,ω) w.r.t. G,μ.

Another notation for the symp. quotient: $M/\!\!/G := (M_{red},\omega_{red})$.

Idea of proof: · for $p\in M$, $d\mu_p: T_pM\to\mathfrak{g}^*$
has (1) $\ker d\mu_p = (T_p\underbrace{\mathcal{O}_p}_{G\text{-orbit through } p})^{\perp}$, $\quad T_p\mathcal{O}_p = \operatorname{im}(\psi|_p:\mathfrak{g}\to T_pM)$

$\omega_p(\psi(X)|_p,-)=d\langle X,\mu_p\rangle$, $\quad X\in\mathfrak{g}$



(2) $\operatorname{im} d\mu_p = \operatorname{Ann}(\underbrace{\operatorname{stab}(p)}_{\subset\mathfrak{g}}) = \{\xi\in\mathfrak{g}^* \mid \langle\xi,X\rangle=0 \ \forall X\in\operatorname{stab}(p)\}$

Corollaries: a) G-action is locally free at p $\iff$ $\operatorname{stab}(p)=0$ $\quad(\operatorname{stab}(p)\subseteq\mathfrak{g})$
$\iff$ $d\mu_p$ is surjective
$\iff$ p is a regular point of μ

b) G acts freely on $\mu^{-1}(0)$ $\Rightarrow$ 0 is a reg. value of μ
$\Rightarrow$ $\mu^{-1}(0)$ is a closed submfd of M of codim $=\dim G$

c) G acts freely on $\mu^{-1}(0)$ $\Rightarrow$ $T_p\mu^{-1}(0) = \ker d\mu_p$ (for $p \in \mu^{-1}(0)$)

$\Rightarrow$ $T_p\mathcal{O}_p$ and $T_p\mu^{-1}(0)$ are symp. orthogonal in T_pM

Proof of (3)

- $\mathcal{O}_p$ is isotropic : $\omega_p(\psi(X)_p, \psi(Y)_p) = \{\langle Y,\mu\rangle, \langle X,\mu\rangle\}_p$

$X_{\langle X,\mu\rangle}$ $X_{\langle Y,\mu\rangle}$ $= \{\mu^*(Y), \mu^*(X)\}_p$

$\underset{\text{equivariance of } \mu}{=} \mu^*([Y,X])_p = \langle [X,Y],\mu\rangle|_p$

$= 0$

$\Rightarrow$ $\mu^{-1}(0) \subset M$ is coisotropic

with $T_p\mathcal{O}_p \subset T_p\mu^{-1}(0)$ the char. distribution ; its leaves (of the corresp. foliation) = G-orbits on $\mu^{-1}(0)$

we have a symp. structure ω_p on $T_p\mu^{-1}(0) / T_p\mathcal{O}_p$ (linear coiso reduction)

$\quad\quad = T_{[p]}M_{red}$ ← class of p in M_{red}

$\omega_{[p]}$ as a symp. form on $T_{[p]}M_{red}$ is well-defined, since ω is G-invariant $\Rightarrow$ we have a non-deg. 2-form ω_{red} on M_{red}

$$\begin{array}{ccc} \mu^{-1}(0) & \xhookrightarrow{i} & M \\ \downarrow \pi & & \\ M_{red} & & \end{array}$$

we have $i^*\omega = \pi^*\omega_{red}$ by construction

$\Rightarrow$ $\pi^*d\omega_{red} = i^*d\omega = 0$ $\underset{\text{by injectivity of } \pi^*}{\Longrightarrow}$ $d\omega_{red} = 0$

– This proves (3) of the Thm.

So: $\mu^{-1}(0)$ is a closed submfd of M with a free action of a cpt group G

$\underset{\text{Thm 23.4 in CdS (general statement about free actions)}}{\Longrightarrow}$ $\mu^{-1}(0)/G$ is a manifold and $\cdot$ $\mu^{-1}(0) \circlearrowleft G$ $\downarrow$ $\mu^{-1}(0)/G$ is a G-bundle

– parts (1) and (2) of the Thm.

$\checkmark$

Example $M = \mathbb{C}^n \simeq T^*\mathbb{R}^n$ (p, q), $\omega = \frac{i}{2}\sum_k dz_k \wedge d\bar{z}_k = \sum_k dq_k \wedge dp_k$ $\quad z_k = q_k + i p_k$

$G = S^1$ acting by $e^{i\theta}\cdot(z_1,\dots,z_n) = (e^{i\theta}z_1,\dots,e^{i\theta}z_n)$

moment map

$$\mu : M \longrightarrow \mathfrak{g}^* \simeq \mathbb{R}$$
$$(z_1,\dots,z_n) \longmapsto -\frac{1}{2}\Big(\sum_k |z_k|^2 - \underbrace{1}_{\text{can choose any constant here}}\Big)$$

$\psi(i\theta)|_z = (i\theta z_1,\dots,i\theta z_n) \in T_z M$

$\iota_{\psi(i\theta)}\omega = -\frac{1}{2}\sum_k (z_k d\bar{z}_k + \bar{z}_k dz_k)$
$= d\left(-\frac{1}{2}\sum_k z_k \bar{z}_k\right)$

$\mu^{-1}(0) = \underset{\text{unit sphere}}{S^{2n-1}} \subset \mathbb{C}^n$

reduction: $\mu^{-1}(0)/G = \dfrac{\{\vec z \in \mathbb{C}^n \mid \|\vec z\| = 1\}}{\vec z \sim e^{i\theta}\vec z} = \dfrac{\mathbb{C}^n \setminus \{0\}}{\vec z \sim \lambda \vec z \;\; \forall \lambda \in \mathbb{C}^*} = \mathbb{C}P^{n-1}$ – reduced space

Rem If $G \circlearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{g}^*$ is Hamiltonian G-space,
and $H \circlearrowright (M,\omega) \xrightarrow[\nu]{} \mathfrak{h}^*$ Ham. action of H on M commuting with G-action and s.t. ν is G-invariant,
then $M_{red} = M/\!/G$ inherits a Ham. action of H with moment map $\nu_{red} : M_{red} \to \mathfrak{h}^*$
satisfying $\nu_{red} \circ \pi = \nu \circ i$
$\mathfrak{h}^* \leftarrow M_{red} \leftarrow \mu^{-1}(0)$ $\quad \mathfrak{h}^* \leftarrow M \leftarrow \mu^{-1}(0)$ or:

$$\begin{tikzcd}
\mu^{-1}(0) \arrow[r, hook, "i"] \arrow[d, "\pi"'] & M \arrow[d, "\nu"] \\
M_{red} \arrow[r, "\nu_{red}"'] & \mathfrak{h}^*
\end{tikzcd}$$

Ex: (Atiyah-Bott, cont'd) for Σ closed oriented surface, $\mathfrak{g} = \mathrm{Lie}(G)$ equipped with ad-invar. nondeg. pairing;

$$\begin{matrix} P \circlearrowleft G \\ \downarrow \\ \Sigma \end{matrix}$$ a principal bundle,

we have $\underset{\mathcal{G}}{\mathrm{Aut}(P)} \circlearrowright (\mathrm{Conn}(P), \omega_{AB}) \xrightarrow{\mu = \mathrm{curv}} \Omega^2(M, \mathrm{ad}(P))$

Symplectic reduction: $\mu^{-1}(0)/\mathcal{G}$ = moduli space of flat connections in P, carries Atiyah-Bott symp. structure

($\mathcal{G}$ does not act freely on Conn $\Rightarrow$ MWM thm does not apply literally
$\rightsquigarrow \mu^{-1}(0)/\mathcal{G}$ has "orbifold singularities")