LAST TIME: Moser's trick: if α_t is a family of k-forms on M

and $\mathcal{L}_{\boxed{X_t}}\alpha_t + \dot{\alpha}_t = 0$ then $\varphi_t = \mathrm{Flow}_0^t\{X_t\}$ is an isotopy with $\varphi_t^*\alpha_t = \alpha_0$.

time-dependent v.field

Things we used when applying Moser's trick: $\iota_{X_t}\underset{\text{vol. form}}{\alpha_t} = \underset{(n-1)\text{-form}}{\beta}$ has a unique sol $\longleftarrow$ for $\alpha \underset{\neq 0}{\in} \mathrm{Hom}(\Lambda^n V, \mathbb{R}), \beta \in \mathrm{Hom}(\Lambda^{n-1}V, \mathbb{R})$

Lin alg $\rightsquigarrow$ $\alpha^{\#} \in \mathrm{Hom}(\Lambda^{n-1}V, V^*) \; -\iota_v \circ$
$= \mathrm{Hom}(V, (\Lambda^{n-1}V)^*)$

$\exists!\ v \in V$ s.t. $\alpha(v, -, \dots, -) = \beta$

$$\boxed{v = (\alpha^{\#})^{-1} \circ \beta}$$

Likewise: $\iota_{X_t}\underset{\text{symp form}}{\omega_t} = \underset{1\text{-form}}{\beta}$ has a unique sol. for X_t.

Thm (Weinstein's Darboux theorem)

Let M be a smooth mfd and $i: N \hookrightarrow M$ a cpt submfd. Let ω_0 and ω_1 be two symp. forms on M s.t. $\omega_0|_N = \omega_1|_N$. Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of N in M and a smooth map $\varphi: \mathcal{U}_0 \to \mathcal{U}_1$ s.t. $\varphi|_N = id$ and $\varphi^*\omega_1 = \omega_0$.

Lemma (version of Poincaré lemma) Let $i: N \hookrightarrow M$ be a submfd, $\alpha \in \Omega^k(M)$ a closed k-form s.t. $i^*\alpha = 0$. Then one can find a nbhd $\mathcal{U}$ of N in M and a $(k-1)$-form $\beta \in \Omega^{k-1}(\mathcal{U})$ with $\beta|_N = 0$ s.t. $\alpha = d\beta$ on $\mathcal{U}$.

Proof of Weinstein's Darboux thm Let $\omega_t = (1-t)\omega_0 + t\omega_1$.

$\omega_0|_N = \omega_1|_N \underset{\text{using compactness of } N}{\Longrightarrow}$ can find a nbhd $\mathcal{U}$ of N s.t. ω_t is symplectic in $\mathcal{U}$.

By Lemma, $\exists\ \alpha \in \Omega^1(\mathcal{U})$ s.t. $\dot{\omega}_t = \omega_1 - \omega_0 = d\alpha$.

Moser's eq (@): $0 = \mathcal{L}_{X_t}\omega_t + \dot{\omega}_t = d(\iota_{X_t}\omega_t + \alpha) \Leftarrow \iota_{X_t}\omega_t = -\alpha$ - solve for X_t

α vanishes on N $\Rightarrow$ X_t vanishes on N $\Rightarrow$ can shrink $\mathcal{U}$ to a sub-nbhd $\mathcal{U}_0$ s.t.

$\varphi_t = \mathrm{Flow}_0^t\{X_t\}$

- is defined on $\mathcal{U}_0$
- is stationary on N
- intertwines ω_0, ω_t.

$\rightsquigarrow$ set $\mathcal{U}_1 = \varphi_1(\mathcal{U}_0)$

□

(usual) **Darboux thm**: Let (M,ω) be a symp. $(2n)$-manifold. Then for any $p \in M$ $\exists\ \underset{\substack{\rotatebox{90}{$\in$}\\ p}}{\mathcal{U}} \subset M$ a nbhd and a nbhd $\underset{\substack{\rotatebox{90}{$\in$}\\ 0}}{\mathcal{U}_0} \subset \mathbb{R}^{2n}$ s.t. $(\mathcal{U},\omega)$ is symplectomorphic to $(\mathcal{U}_0, \underset{\uparrow}{\omega_0})$

stand. symp. form on $\mathbb{R}^{2n}$.

(i.e. for any $p \in M$ $\exists$ a coord. patch $(\mathcal{U}, x_1 .. x_n, y_1, .. y_n)$ centered at p s.t. on $\mathcal{U}$, $\omega = \sum_i dx_i \wedge dy_i$)

Proof Let $\psi : \underset{\substack{\rotatebox{90}{$\in$}\\ p}}{\overset{\substack{M\\ \cup}}{\mathcal{U}'}} \to \mathbb{R}^{2n}$ some coord. chart on a nbhd $\mathcal{U}'$ of M

s.t. $\omega_p = \sum_i (dx_i \wedge dy_i)_p$ - at p

(can achieve it by doing a linear transformation of $\mathbb{R}^{2n}$, by the normal form thm for a skew-sym. bilinear form)

Thus: we have two symp forms on $\mathcal{U}'$,
ω and $\psi^*\omega_0$ and they agree at p.

Apply Weinstein's Darboux thm for $\{p\} \subset M$ $\Rightarrow$ $\exists\ \underset{\substack{\rotatebox{90}{$\in$}\\ p}}{\mathcal{U}}, \mathcal{U}_1 \subset \mathcal{U}'$ and $\varphi : \mathcal{U} \to \mathcal{U}_1$

s.t. $\varphi(p) = p$,
$\omega = \underbrace{\varphi^* \psi^*}_{(\psi\circ\varphi)^*} \omega_0$

$\psi \circ \varphi : \underset{\substack{\cap\\ M}}{\mathcal{U}} \longrightarrow \mathcal{U}_0 = \psi(\mathcal{U}_1) \subset \mathbb{R}^{2n}$

- the Darboux chart □.

Weinstein's Lagrangian neighborhood theorem

Let M be $(2n)$-mfd, $i: X \hookrightarrow M$ a cpt n-dim. submfd. Let ω_0, ω_1 be two symp. forms on M s.t. $i^*\omega_0 = i^*\omega_1 = 0$ (i.e. X is Lagrangian w.r.t. ω_0 and ω_1).

Then there exist neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of X in M and a diffeo $\varphi: \mathcal{U}_0 \to \mathcal{U}_1$ s.t. $\varphi^*\omega_1 = \omega_0$.

– relies on

"Whitney extension theorem" Let M be an m-mfd, $X \subset M$ a k-dim. submfd, $k<m$. Suppose $\forall x \in X$ we are given $\lambda_p : T_pM \xrightarrow{\sim} T_pM$ – a linear iso smoothly depending on $p \in X$ satisfying $\lambda_p \big|_{T_pX} = id$.

Then there exists an embedding $h: \mathcal{N} \to M$ of some nbhd $\mathcal{N}$ of X in M s.t.

$h|_X = id_X$ and $dh_p = \lambda_p \quad \forall p \in X$.

(I.e. λ_p's determine a linear approx. (1-jet) of the embedding on X)

Need the following from symp. lin. alg.

Lemma 1 Let (V, B) be a $(2n)$-dim. symp. v.sp., $L \subset V$ a Lagrangian subspace, $W \subset V$ some (not necessarily Lagrangian) complement of L in V.

Then from W we can canonically build a Lagr. complement W' of L.

Proof B induces a nondeg. pairing $B': L \times W \to \mathbb{R}$. $\Rightarrow (B')^\#: L \xrightarrow{\sim} W^*$ is an iso

Let $A \in \mathrm{Hom}(W, L)$, $W' = \mathrm{graph}(A) = \{w + Aw \mid w \in W\}$. When is W' Lagrangian?

$\forall w_1, w_2 \in W$, $0 \overset{\text{WANT}}{=} B(w_1 + Aw_1, w_2 + Aw_2) = B(w_1,w_2) + B(Aw_1, w_2) + B(w_1, Aw_2) + \underbrace{B(\underbrace{Aw_1, Aw_2}_{\in L})}_{0}$

$\Rightarrow B(w_1, w_2) \overset{\text{WANT}}{=} B(Aw_2, w_1) - B(Aw_1, w_2)$
$= B^\#(Aw_2)(w_1) - B^\#(Aw_1)(w_2)$

Let $A' = B'^\# \circ A \in \mathrm{Hom}(W, W^*)$

$\Rightarrow$ we are looking for A' s.t. $\forall w_1, w_2 \in W$, $B(w_1, w_2) = A'(w_2)(w_1) - A'(w_1)(w_2)$.

Canonical choice: $A'(w) = -\frac{1}{2} B(w, -)$, $A = (B'^\#)^{-1} \circ A'$ $\square$

Lemma 2 Let V be a $2n$-dim. v.space with B_0, B_1 two symp. forms on V. Let $L \subset V$ be a subspace Lagrangian w.r.t. both B_0 and B_1 and let W be any complement of L in V. Then from W one can canonically construct a linear iso $\lambda : V \xrightarrow{\sim} V$ s.t. $\lambda|_L = id_L$ and $\lambda^* B_1 = B_0$.

Proof By Lemma 1: $W \rightsquigarrow$ complement W_0 of L, Lagr. wrt B_0

$W \rightsquigarrow$ —— W_1 —— B_1

nondeg. pairings $W_0 \times L \xrightarrow{B_0} \mathbb{R}$, $W_1 \times L \xrightarrow{B_1} \mathbb{R}$ $\rightsquigarrow$ isomorphisms $B_0^\# : W_0 \xrightarrow{\sim} L^*$, $B_1^\# : W_1 \xrightarrow{\sim} L^*$

Let $\Lambda = (B_1^\#)^{-1} \circ B_0^\# : W_0 \xrightarrow{\sim} W_1$. It satisfies $B_1^\# \circ \Lambda = B_0^\#$

i.e. $B_1(\Lambda w_0, u) = B_0(w_0, u)$ $\forall\, w_0 \in W_0, u \in L$.

Extend Λ to V as $\lambda := Id_L \oplus \Lambda : L \oplus W_0 \longrightarrow L \oplus W_1$

Then: $\lambda^* B_1 (u + w_0, u' + w_0') = B_1(u + \Lambda w_0, u' + \Lambda w_0') =$

$$= \underbrace{B_1(u,u')}_{0} + \underbrace{B_1(\Lambda w_0, u')}_{B_0(w_0,u')} + \underbrace{B_1(u, \Lambda w_0')}_{B_0(u,w_0')} + \underbrace{B_1(\Lambda w_0, \Lambda w_0')}_{0} = B_0(u + w_0, u' + w_0')$$

$\Rightarrow L$ intertwines B_0 and B_1 $\square$

Proof of W.L.n.thm:

Choose g - a Riem. metric on M. Fix $p \in X$, let $V = T_pM$, $U = T_pX$, $W = U^\perp$

metric complement wrt g_p

$U \subset V$ is a Lagr. subspace w.r.t. both $\omega_0|_p$ and $\omega_1|_p$ (since X is Lagrangian w.r.t. both)

$U^\perp \underset{\text{Lemma 2}}{\rightsquigarrow}$ lin. iso $\lambda_p : T_pM \longrightarrow T_pM$ s.t. $\lambda_p|_{T_pX} = id$ and $\lambda_p^*(\omega_1)_p = (\omega_0)_p$.

– it varies smoothly with p since the construction was canonical

$\underset{\text{Whitney extension thm}}{\rightsquigarrow}$ $\exists$ $\underset{X}{\overset{}{N}} \overset{}{\subset} M$ a nbhd and $h : N \hookrightarrow M$ an embedding s.t. $h|_X = id_X$ and $dh_p = \lambda_p$ $\forall p \in X$

($N \supset X$)

$$\Rightarrow \forall p \in X, \quad (h^*\omega_1)_p = (dh_p)^*(\omega_1)_p = L_p^*(\omega_1)_p = (\omega_0)_p$$

$\Rightarrow$ apply Weinstein's Darboux thm to $\omega_0, h^*\omega_1$

$\exists\ \underset{\overset{\cup}{X}}{\mathcal{U}_0} \subset M$ a nbhd and an embedding $f: \mathcal{U}_0 \hookrightarrow N$

s.t. $f|_X = id_X$ and $f^*(h^*\omega_1) = \omega_0$ on $\mathcal{U}_0$

Set $\varphi = h \circ f$

$\square$

Weinstein's tubular neighborhood thm

Let (M,ω) be a symp. mfd, $i: X \hookrightarrow M$ a Lagrangian submfd, ω_0 the canonical symp. form on T^*X, $i_0: X \hookrightarrow T^*X$ the Lagrangian embedding as a zero-section. Then $\exists$ neighborhoods $\mathcal{U}_0$ of X in T^*X, $\mathcal{U}$ of X in M and a diffeo $\varphi: \mathcal{U}_0 \to \mathcal{U}$ s.t.

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U} \\ & X \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

commutes and $\varphi^*\omega = \omega_0$.