LAST TIME:

Thm (Weinstein's Darboux theorem)

Let M be a smooth mfd and $i: N \hookrightarrow M$ a cpt submfd. Let ω_0 and ω_1 be two symp. forms on M s.t. $\omega_0|_N = \omega_1|_N$. Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of N in M and a smooth map $\varphi: \mathcal{U}_0 \to \mathcal{U}_1$ s.t. $\varphi|_N = id$ and $\varphi^* \omega_1 = \omega_0$.

Thm (Weinstein's Lagrangian neighborhood thm)

Let M be a mfd, ω_0, ω_1 two symp. structures on M, $X \subset M$ a submfd which is Lagr. wrt ω_0 and ω_1. Then $\exists$ $\mathcal{U}_0, \mathcal{U}_1$ two nbhds of X in M and a diffeo

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

s.t. $\varphi^* \omega_1 = \omega_0$.

Rem For (V, B) a symp. v.space and $L \subset V$ a Lagrangian, B induces a non-degenerate pairing $B' : L \times V/L \to \mathbb{R}$

Cor: If $X \subset (M, \omega)$ a Lagrangian, then $N_x X \underset{\omega_x}{\cong} T_x^* X$

T_xM / T_xX ← Lag subspace of T_xX

⇝ there is a can. isomorphism of bundles

$$NX \underset{\omega}{\cong} T^*X$$

(Standard) Tubular neighborhood theorem

$(i^*TM)/TX$

Let M be an m-dim. mfd, $i : X \hookrightarrow M$ a k-dim submfd, $NX = (i^*TM)/TX$ the normal bundle of X in M, $i_0 : X \hookrightarrow NX$ the zero-section. Then there are neighborhoods $\mathcal{U}_0$ of X in NX and $\mathcal{U}$ of X in M and a diffeo $\varphi : \mathcal{U}_0 \to \mathcal{U}$

s.t.

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U} \\ & X \arrow[ul, hook, "i_0"] \arrow[ur, hook, "i"'] & \end{tikzcd}$$

commutes.

Weinstein's tubular neighborhood thm

Let (M, ω) be a symp. mfd, $i : X \hookrightarrow M$ a Lagrangian submfd, ω_0 the canonical symp. form on T^*X, $i_0 : X \hookrightarrow T^*X$ the Lagrangian embedding as a zero-section. Then $\exists$ neighborhoods $\mathcal{U}_0$ of X in T^*X, $\mathcal{U}$ of X in M and a diffeo $\varphi : \mathcal{U}_0 \to \mathcal{U}$ s.t.

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U} \\ & X \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

commutes and $\varphi^* \omega = \omega_0$.

Proof 1. $NX \simeq T^*X$

$\underset{\text{By tub. nbhd thm.}}{\overset{\text{usual}}{\Longrightarrow}}$ $\exists$ nbhd $\mathcal{N}_0 \subset T^*X$ and nbhd $\mathcal{N} \subset M$ and diffeo $\psi : \mathcal{N}_0 \to \mathcal{N}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \cup \quad\quad\quad\quad\quad\quad\quad\quad \cup$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad X \quad\quad\quad\quad\quad\quad\quad\quad X$

s.t.
$$\begin{tikzcd} \mathcal{N}_0 \arrow[rr, "\psi"] & & \mathcal{N} \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd}$$
commutes.

Let $\left.\begin{array}{l} \omega_0 = \text{can. symp form on } T^*X \\ \omega_1 = \psi^*\omega \end{array}\right\}$ symp forms on $\mathcal{N}$

X is Lagr. wrt ω_0 and ω_1

$\underset{\text{Weinstein's Lag nbhd thm}}{\Longrightarrow}$ $\exists$ nbhds $\mathcal{U}_0, \mathcal{U}_1$ of X in $\mathcal{N}_0$ and a diffeo $\theta : \mathcal{U}_0 \to \mathcal{U}_1$ s.t.

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\theta"] & & \mathcal{U}_1 \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd}$$
commutes and $\omega_0 = \theta^*\omega_1 \left(= \underbrace{\theta^*\psi^*}_{(\psi\circ\theta)^*}\omega \right)$

Set $\varphi = \psi \circ \theta$ $\quad\quad \square$

(S. 9.3 in Cannas da Silva)

Application

$\mathrm{Sympl}(M,\omega) = \{\varphi : M \xrightarrow{\simeq} M\}$ symplectomorphisms
- group

What is $\underset{\text{tangent space}}{T_{id}}\, \mathrm{Sympl}(M,\omega)$?
(The Lie algebra of the group of symp-morph)

- C^0, C^1 topology on $\mathrm{Map}(X,Y)$: (X, Y mfds)
  a seq. of maps $f_i : X \to Y$ converges to $f : X \to Y$ in C^0 topology iff f_i converges uniformly on compact sets
  seq. f_i converges to f in C^1 topology if $f_i \to f$ and $\underset{TX \to TY}{df_i \to df}$ converge uniformly on cpt sets

Let (M,ω) cpt. symplectic

for $f \in \mathrm{Sympl}$, $\quad \mathrm{graph}(f) \subset \bar{M} \times M$
$\quad\quad\quad\quad\quad \mathrm{graph}(id) = \Delta \subset \bar{M} \times M$ Lagrangians

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ diagonal

Weinstein tub nbhd thm
$\Rightarrow \exists$ nbhd $\mathcal{U}$ of $\underset{\simeq M}{\Delta}$ in $\bar{M} \times M$ s.t. $\mathcal{U}$ is symplectomorphic to a nbhd $\mathcal{U}_0$ of $M \subset (T^*M, \omega_0)$

$\varphi : \mathcal{U} \to \mathcal{U}_0$
$\quad (p,p) \mapsto (p,0)$

if f "sufficiently C^1-close" to id, $\mathrm{graph}(f) \subset \mathcal{U}$ $\rightsquigarrow$ $\varphi \circ \mathrm{graph}(f) \subset T^*M$ is a projectable Lagrangian

$\Rightarrow \varphi \circ \mathrm{graph}(f) = \mathrm{graph}(\mu)$ $\quad$ in C^1-topology
$\quad\quad$ for some $\mu \in \Omega^1_{\text{closed}}(M)$

So: $\exists$ a neighborhood $\mathcal{S}$ of $id : M \to M$ in $\mathrm{Sympl}(M,\omega)$ and a neighborhood $\mathcal{F}$ of of $0 \in \Omega^1_{\text{closed}}(M)$ and a homeomorphism $\mathcal{S} \xrightarrow{\Phi} \mathcal{F}$

$f \longmapsto \mu$ s.t. $\varphi \circ \mathrm{graph}(f) = \mathrm{graph}(\mu)$

Thus: $T_{id}\,\mathrm{Sympl}(M,\omega) \simeq \Omega^1_{closed}(M)$

In particular $T_{id}\,\mathrm{Sympl}$ contains $\{\mu = dh \mid h \in C^\infty(M)\} \simeq C^\infty(M)/_{\text{loc. constant functions}}$

$= \Omega^1_{exact}(M)$

<u>Cor</u>: Let (M,ω) be cpt symplectic mfd with $H^1_{dR}(M)=0$.

Then any symplectomorphism $f \in \underset{C^1\text{-neighborhood of } id}{S}$ has at least two fixed points

<u>Proof</u> $\underset{S}{f \in} \xleftrightarrow{\Phi} \mu \in \Omega^1_{cl}(M) \underset{\text{since } H^1(M)=0}{=} \Omega^1_{ex}(M) \Rightarrow \mu = dh$ h has at least two extrema (using compactness)

$\Rightarrow \mu$ has at least two zeros.

$\Rightarrow f$ has at least two fixed points $\square$

<u>Cor</u> Let (M,ω) be a symp. mfd, $X \subset M$ a cpt Lagrangian with $H^1_{dR}(X)=0$.

Then every Lagrangian Y in M which is C^1-close to X intersects X in at least two points.

$\exists$ diffeo $X \xrightarrow{\simeq} Y \hookrightarrow M$ which is C^1-close to $X \xhookrightarrow{i} M$

Arnold's conjecture

Let (M,ω) cpt symp. mfd, $f = \mathrm{Flow}_0^1\{X_t\}$ — time-dependent v.field on M satisfying $\iota_{X_t}\omega = dh_t$ for $h_t \in C^\infty(S^1 \times M)$ -periodic family of functions on M

a symplectomorphism

Then: $\#\{\text{nondeg. fixed points of } f\} \geq \sum_{i=0}^{2n} \dim H^i(M;\mathbb{R})$

⟨proven by Floer et al.⟩

$df_p : T_pM \to T_pM$ is an iso

More precisely: <u>Arnold's conjecture</u>:

$\#\{\text{fixed pts of } f\} \geq$ minimal # of crit. points a smooth function F on M can have $\underbrace{\geq}_{\text{by Morse theory}} \sum_{i=0}^{2n} \dim H^i(M,\mathbb{R})$

assuming: F is Morse, f has nondeg. cr. pts

Hamiltonian vector fields

For (M, ω) a symp. mfd and $H \in C^\infty(M)$ a function,

$\exists!$ v. field X_H on M s.t. $\boxed{\iota_{X_H} \omega = dH}$

def X_H - the "Hamiltonian vector field" with 'Hamiltonian function H.

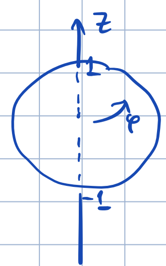
Hamiltonian flow

- Let $\rho_t = \mathrm{Flow}_t(X_H) : M \to M$ (suppose that M is cpt or if not, that X_H is complete)

Then $\frac{d}{dt} \rho_t^* \omega = \rho_t^* \underbrace{\mathcal{L}_{X_H} \omega}_{d(\iota_{X_H}\omega) = d^2 H = 0} = 0 \underset{\text{knowing } \rho_0^* \omega = \omega \text{ since } \rho_0 = id}{\Longrightarrow} \boxed{\rho_t^* \omega = \omega}$

Thus, X_H gives rise to a 1-parameter family of symplectomorphisms of M.

Ex: $M = S^2$ with coords φ, z, $\omega = d\varphi \wedge dz$



$H = z$ (height function) $\rightsquigarrow$ $\iota_{X_H} \omega = dH \iff X_H = \frac{\partial}{\partial \varphi}$

Thus $\rho_t(\varphi, z) = (\varphi + t, z)$ rotation about the vertical axis

Note: z is preserved by this motion.

Generally: H is preserved along orbits of $\rho_t = \mathrm{Flow}_t X_H$. ($\leftrightarrow$ each integral curve of X_H ($\{\rho_t(x) \mid t \in \mathbb{R}\}$) is contained in a level set of H)

Indeed: $\frac{d}{dt} \rho_t^* H = \rho_t^* \underbrace{\mathcal{L}_{X_H} H}_{\iota_{X_H} dH = \iota_{X_H} \iota_{X_H} \omega = 0} = 0$.

<u>Ex:</u> $M = \mathbb{R}^{2n}$ with coords $q_1, \dots, q_n, p_1, \dots, p_n$, $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$; Let $H \in C^\infty(\mathbb{R}^{2n})$

$$\iota_{X_H} \omega = dH = \sum_i^n \frac{\partial H}{\partial q_i} dq^i + \frac{\partial H}{\partial p_i} dp_i \iff X_H = \sum_i \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i}$$

a curve $(q(t), p(t))$ is an integral curve of X_H iff

$$\begin{cases} \dfrac{dq_i(t)}{dt} = \dfrac{\partial H(q,p)}{\partial p_i} \\[2ex] \dfrac{\partial p_i(t)}{dt} = -\dfrac{\partial H(q,p)}{\partial q_i} \end{cases}$$

- Hamilton's equations

Example from class. mechanics

E.g. $n = 3$, $M = T^*\mathbb{R}^3$

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + V(q)$$

$V \in C^\infty(\mathbb{R}^3)$ force potential

Hamilton's eqs:

$$\left.\begin{aligned} \dot{q}_i &= \frac{1}{m} p_i \\ \dot{p}_i &= -\partial_i V(q) \end{aligned}\right\} \Rightarrow m\ddot{\vec{q}} = -\nabla V(q)$$

- Newton's equation of motion of particle of mass m in a force field with potential $V(q)$

$\vec{q}$ = position

$\vec{p}$ = momentum

H = energy function

M = phase space