Hamiltonian torus actions & convexity theorems

Thm (Atiyah-Guillemin-Sternberg)

Let (M,ω) be a compact connected symp. mfd and let $\underset{G}{\mathbb{T}^m} = \mathbb{R}^m/\mathbb{Z}^m$ be an m-torus.

Let Ψ be a Hamiltonian action of $\mathbb{T}^m$ on M with equivariant moment map $\mu: M \to \mathbb{R}^m = \mathfrak{g}^*$.

Then:

(1) $\underbrace{\mathrm{Fix}_{\mathbb{T}^m}(M)}_{\text{fixed points of } \mathbb{T}^m\text{-action}} = \coprod_{j=1}^{N} C_j$ - finite union of symplectic submanifolds of M

(2) Level sets of μ are connected

(3) The image of μ is convex

(4) The image of μ is the convex hull of images of fixed points of $\mathbb{T}^m$-action.

- The image $\mu(M)$ of the moment map is called the "moment polytope"

polytope in $\mathbb{R}^n$ = convex hull of fin. many points in $\mathbb{R}^n$

convex polyhedron in $\mathbb{R}^n$ = intersection of finitely many affine half-spaces

so: polytope = bounded convex polyhedron

Ex: $M = \mathbb{C}P^n$ $\quad \omega$ = symp. form arising from symp reduction $\mathbb{C}P^n = \mathbb{C}^{n+1}/\!\!/S^1$ with const

$= \frac{i}{2}\partial\bar\partial \log(|z_0|^2 + \dots + |z_n|^2)$

action $\mathbb{T}^n \circlearrowright \mathbb{C}P^n$ by $\Psi\big((e^{i\theta_1},\dots,e^{i\theta_n}),(z_0:z_1:\dots:z_n)\big)$

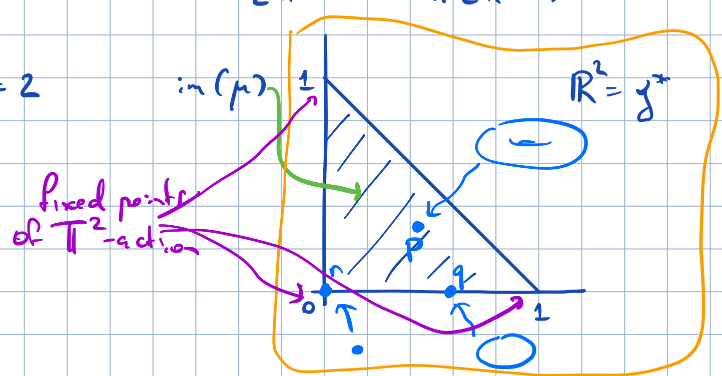
$= (z_0 : e^{i\theta_1}z_1 : \dots : e^{i\theta_n}z_n)$

moment map:

$\mu: \mathbb{C}P^n \to \mathbb{R}^n$

$(z_0:\dots:z_n) \mapsto \left(\frac{|z_1|^2}{\|\vec z\|^2}, \dots, \frac{|z_n|^2}{\|\vec z\|^2}\right)$ $\quad \leftarrow$ scale the stand. symp. form by (-2) so as to avoid a factor $(-\frac{1}{2})$ here.

E.g. $n=2$



for $\overset{(x_1,x_2)}{p}$ inside the triangle,

$\mu^{-1}(p) = $ 2-torus $\{(1 : e^{i\theta_1}\sqrt{x_1} : e^{i\theta_2}\sqrt{x_2})\}$

for q on the boundary (not corner),

$\mu^{-1}(q) = S^1$ $\quad$ (e.g. $(1 : e^{i\theta}\sqrt{x} : 0)$ if $q = (x,0)$

or $(0 : \sqrt{x} : e^{i\theta}\sqrt{1-x})$ if $q = (x, 1-x)$)

for r a corner (vertex),

$\mu^{-1}(r) = pt$

stabilizers $\mathbb{CT}^2$: $\quad \mathrm{Stab}(p) = \{1\}$, $\mathrm{Stab}\{q\} = S^1$, $\mathrm{Stab}\{r\} = \mathbb{T}^2$

Rem One can view this example like this:

$$\begin{array}{ccc} S^1 \overset{\Phi_1}{\circlearrowright} & & \xrightarrow{\mu_1} \mathbb{R} \\ & \mathbb{C}^{n+1} & \\ \mathbb{T}^n \underset{\Phi_2}{\circlearrowright} & & \xrightarrow[\mu_2]{} \mathbb{R}^n \end{array}$$

Commuting ham. actions

Φ_1: $e^{i\theta} \cdot (z_0, \dots, z_n) = (e^{i\theta} z_0, \dots, e^{i\theta} z_n)$, $\quad \mu_1(z_0, \dots, z_n) = \|\vec{z}\|^2 - 1$

- diagonal action

Φ_2: $(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot (z_0, \dots, z_n) = (z_0, e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$, $\quad \mu_2(z_0, \dots, z_n) = (|z_1|^2, \dots, |z_n|^2)$

Then: $\mathbb{CP}^n = \mathbb{C}^n /\!/ S^1$ - inherits the ham. action $\underline{\Phi}_2$ with moment map $\underline{\mu_2}$.

- A group action $G \circlearrowright M$ is called "effective" ("faithful") if each $g \in G$ moves at least one point in M (i.e. no elt. of G acts as identity)

<u>Fact</u> any effective torus action $\mathbb{T}^m \circlearrowright M$ (not necessarily Hamiltonian) has orbits of dimension m.

<u>Corollary</u> (of AGS theorem) Under conditions of AGS thm., if the action $\mathbb{T}^m \circlearrowright M$ is effective, then there must be at least $m+1$ fixed points.

Proof: $\mathbb{T}^m$ effective $\Rightarrow \exists$ a point $p \in M$ where μ is a submersion, i.e. $(d\mu_1)_p, \dots, (d\mu_m)_p$ are lin. indep.

$\Rightarrow \mu(p)$ is an interior point of $\operatorname{im}(\mu) \underset{AGS}{\Rightarrow} \operatorname{im}(\mu)$ is a non-degenerate convex polytope.

Any nondeg. convex polytope in $\mathbb{R}^m$ has $\geq m+1$ vertices ; vertices of $\operatorname{im}(\mu)$ are images of fixed points $\square$

<u>Prop.</u> Let $(M, \omega, \mathbb{T}^m, \mu)$ be a Ham. $\mathbb{T}^m$-space. If the $\mathbb{T}^m$-action is <u>effective</u>, then $\dim M \geq 2m$.

<u>Proof</u>: (1) $\mathbb{T}^m$-orbits are isotropic in M (see the proof of MWM thm)

(2) since the action is effective, there are orbits of dimension $m = \dim G$.

(1) + (2) $\Rightarrow$ $m \leq \frac{1}{2} \dim M$ $\square$

<u>Def</u> A (symplectic) <u>toric manifold</u> is a cpt connected symplectic mfd (M, ω) equipped with an <u>effective</u> hamiltonian action of a torus $\mathbb{T}^m$ with $\boxed{m = \frac{1}{2} \dim M}$ and with a choice of moment map μ.

<u>Rem</u> a toric manifold gives rise to an integrable system on M with (Poisson-commuting) Hamiltonians $\mu_1, \dots, \mu_m$.
with $\mu = (\mu_1, \dots, \mu_m)$

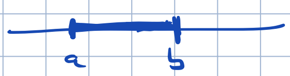
Classification of symplectic toric manifolds

def A Delzant polytope Δ in $\mathbb{R}^n$ is a convex (compact) polytope that is:

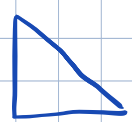
- "simple" – there are exactly n edges meeting at each vertex;
- "rational" – edges meeting at p have the form $p + t u_i$, $t \geq 0$ with $u_i \in \mathbb{Z}^n$;
- "smooth" – for each vertex p, corresponding u_i can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

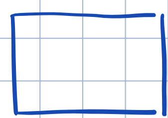
(positions of vertices don't have to be rational)

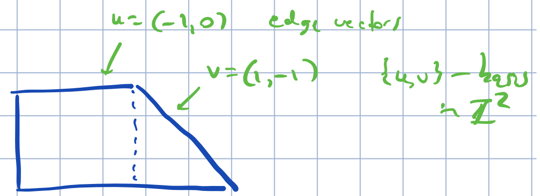
Examples: $n=1$

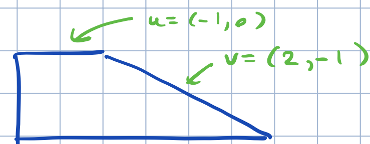


$n=2$

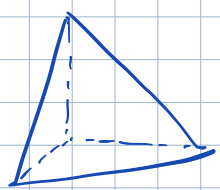






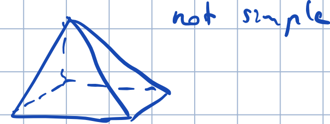


$n=3$



Non-examples $n=2$

$n=3$



Another description of Delzant polytopes

Let Δ be a Delzant polytope in $\mathbb{R}^n$ with d faces (of codim $=1$).

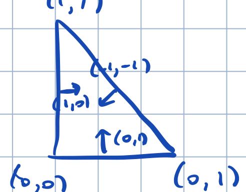
Let $v_i \in \mathbb{Z}^n$, $i=1\dots d$ the primitive inward-pointing normal vectors to faces

$$v \neq k u \text{ for } u \in \mathbb{Z}^n,\ k \geq 2,\ k \in \mathbb{Z}$$

Then: $\{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \geq \lambda_i,\ i=1\dots d\}$ for some $\lambda_i \in \mathbb{R}$.

~~Eucl. inner product on $\mathbb{R}^n$~~

Ex:



$$\Delta = \{x \in (\mathbb{R}^2)^* \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\}$$

$$= \{x \in (\mathbb{R}^2)^* \mid \langle x, (1,0)\rangle \geq 0,\ \langle x, (0,1)\rangle \geq 0,\ \langle x, (-1,-1)\rangle \geq -1\}$$

Delzant theorem

Toric manifolds are classified by Delzant polytopes:
there is a bijection

$$\{\text{toric manifolds}\} \xleftrightarrow{1-1} \{\text{Delzant polytopes}\}$$
$$(M^{2n}, \omega, \mathbb{T}^n, \mu) \longmapsto \mu(M)$$

Sketch of proof of existence (surjectivity)

$\Delta \subset (\mathbb{R}^n)^*$ Delzant with d faces $\xrightarrow{?} (M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu)$

· write $\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \geq \lambda_i, \; i = 1 \ldots d\}$

$v_i \in \mathbb{Z}^n$ primitive inward face vectors.

Let $\pi: \mathbb{R}^d \to \mathbb{R}^n$
$e_i \mapsto v_i$
stand. basis vectors

· Claim: π is onto and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$

idea: (at a vertex P, edge vectors $\{u_i\}$ form a basis of $\mathbb{Z}^n$; $\{v_i\}$ - form a dual basis of $\mathbb{Z}^n$) for incident faces

Thus π induces a surjective map of tori:

$$0 \to \underset{\ker \pi}{\overset{\|}{H}} \xhookrightarrow{i} \underset{\mathbb{R}^d/\mathbb{Z}^d}{\overset{\|}{\mathbb{T}^d}} \xrightarrow{\pi} \underset{\mathbb{R}^n/\mathbb{Z}^n}{\overset{\|}{\mathbb{T}^n}} \to 0 \qquad \text{LES of Lie groups (tori)}$$

$$0 \to \mathfrak{h} \xhookrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \to 0 \qquad \text{LES of Lie alg.}$$

$$0 \leftarrow \mathfrak{h}^* \xleftarrow{i^*} (\mathbb{R}^d)^* \xleftarrow{\pi^*} (\mathbb{R}^n)^* \leftarrow 0 \qquad \text{dual LES}$$

Consider $\mathbb{C}^d$, $\omega_0 = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k$ with stand. ham action of $\mathbb{T}^d$

$$(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_d}) \cdot (z_1, \ldots, z_d) = (e^{2\pi i \theta_1} z_1, \ldots, e^{2\pi i \theta_d} z_d)$$

moment map $\varphi(z_1, \ldots, z_d) = (|z_1|^2, \ldots, |z_d|^2) + C$ (constant) set $C = (\lambda_1, \ldots, \lambda_d)$

Q: what is the moment map for the action restricted to sub-torus H?

A: it is $\underbrace{i^* \circ \varphi}_{\mu_H}: \mathbb{C}^d \to \mathfrak{h}^*$

Let $Z = \mu_H^{-1}(0)$.

Claim: · $0 \in \mathfrak{h}^*$ is a regular value of μ_H
· Z is a cpt submfld of dim $= 2d - \underbrace{(d-n)}_{\dim \mathfrak{h}} = d+n$ and H acts freely on Z.

5

Using MWM thm:
Symplectic quotient:

$$\mathbb{C}^d /\!/ H = Z/H =: M_\Delta$$ is a symplectic mfd, with symp form ω_Δ

orbit space

dim $= (d+n) - (d-n) = 2n$

$$\begin{tikzcd} Z \arrow[r, hook, "j"] \arrow[d, "p"] & \mathbb{C}^d \\ M_\Delta, \omega_\Delta & \end{tikzcd}$$

$Z \circlearrowleft H$ ← H-bundle

$$p^*\omega_\Delta = j^*\omega_0$$

- $\varphi : \mathbb{C}^d \to (\mathbb{R}^d)^*$ induces a moment map

$\mu : M_\Delta \to \ker(i^*) = (\mathbb{R}^n)^*$

Claim: $\mu(M_\Delta) = \Delta$.

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