LAST TIME

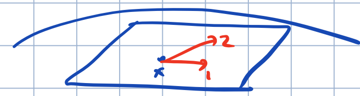
Principal G-bundle

$$\begin{array}{c} P \circlearrowleft G \\ \downarrow \pi \\ M \end{array}$$

- free action, transitive on π-fibers

Ex: Frame bundle of a mfd M : FM

fiber over $x = \{\text{frames in } T_xM\}$.
(ordered bases)



$$\begin{array}{l} FM \circlearrowleft GL(n,\mathbb{R}) \\ \downarrow \qquad\qquad \uparrow \dim M \\ M \end{array}$$

more generally, for E a rank $=k$ V.B.,
one has the frame bundle $FE \circlearrowleft GL(k)$
$\downarrow$
M

Ex: Hopf bundle (Hopf fibration)

$$S^3 = \{(z_0,z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\} \circlearrowleft S^1$$
$$(z_0,z_1) \mapsto (e^{i\theta}z_0, e^{i\theta}z_1)$$

$\downarrow$ line through (z_0,z_1)

$\mathbb{CP}^1 \underset{\text{diffeo}}{\sim} S^2$

Sections vs. trivializations

Given an equivar. trivialization $U\times G \overset{\circlearrowleft G}{\xrightarrow[\varphi]{}} \overset{\circlearrowleft G}{\pi^{-1}(U)}$, one constructs a local section
$\sigma|_U : x \longmapsto \varphi(x, \underset{G}{\underset{\cap}{1}})$. Conversely, given a local section, one constructs a
an equiv. trivialization $\varphi(x,g) = \sigma(x)\cdot g$

- $\exists$ a global section $\Leftrightarrow$ principal bundle is trivial

[This is specific to principal bundles; it is not true e.g. for vector bundles]

Transition functions:

$\{(U_\alpha, \sigma_\alpha)\}$ - loc. triv.
$\qquad\uparrow$
loc. sections

$\sigma_\beta(x) = \sigma_\alpha(x)\cdot t_{\alpha\beta}(x) \qquad x \in U_\alpha \cap U_\beta$

$t_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ transition functions.

Rem:

$$\begin{tikzcd}
 & \pi^{-1}(U_\alpha\cap U_\beta) & \\
(U_\alpha\cap U_\beta)\times G \arrow[ur, "\varphi_\alpha"] & & (U_\alpha\cap U_\beta)\times G \arrow[ul, "\varphi_\beta"']
\end{tikzcd}$$

$(x,g) \mapsto \sigma_\alpha(x)\cdot g \overset{"}{=} \sigma_\beta(x)\,t_{\beta\alpha}(x)\cdot g \mapsto (x, \underbrace{t_{\beta\alpha}(x)\cdot g}_{})$

so, transition functions act from the left (as usual)

Associated bundle construction

If $\begin{matrix} P \circlearrowleft G \\ \downarrow \pi \\ M \end{matrix}$ - principal bundle and $\rho: G \to GL(V)$ a linear representation of G on a v.sp. V,

then $E = P \times_G V = \{(p,v) \in P \times V\} \big/ (p \cdot g, v) \sim (p, \rho(g)(v)) \ \forall g \in G$

is the "associated vector bundle".

Note: projection $\begin{matrix} E & (p,v) \\ \downarrow & \mapsto \\ M & \pi(p) \end{matrix}$ is well-defined.

Exercise: check that for transition functions, one has $t^E_{\alpha\beta} = \rho(t^P_{\alpha\beta})$

More generally: if $\begin{matrix} E \leftarrow F \\ \downarrow \\ M \end{matrix}$

- fiber bundle with transition functions $t_{\alpha\beta}: U_\alpha \cap U_\beta \to \underset{\text{str. group}}{G}$ $\quad G \circlearrowright \underset{\text{fiber}}{F}$

and F' - another space equipped with G-action,

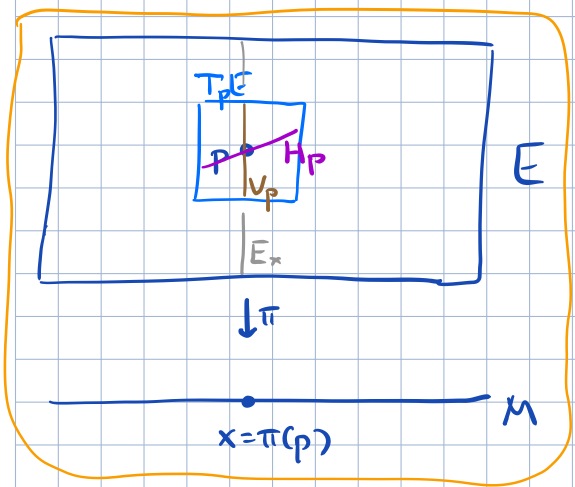
one can construct an associated bundle $\begin{matrix} E' \leftarrow \underbrace{F'}_{\text{new fiber}} \\ \downarrow \\ M \end{matrix}$ with the same transition functions $t_{\alpha\beta}$.

Connections

def An (Ehresmann) connection in a fiber bundle $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$ is a subbundle $H \subset TE$ such that

$$\boxed{H \oplus \underbrace{V}_{\ker(d\pi : TE \to TM)} = TE}.$$

← tangent space to the fiber



I.e., it is a choice of a splitting of

$$\begin{tikzcd} V \arrow[r, hook] & TE \arrow[r, "d\pi"] & TM \arrow[l, bend left, dashed] \end{tikzcd}$$

Note: V is canonical (vertical distribution), whereas H is a choice (of a horizontal distribution)

- projection $\mathcal{A} : TE \to V$ along H can be seen as a 1-form $\mathcal{A} \in \Omega^1(E, V)$ satisfying $\mathcal{A}_p(v) = v$ for $v \in V_p$.
  (i.e. $\mathcal{A}$ is identity on V)

- Curvature (of the connection)

2-form $\mathcal{F} \in \Omega^2(E, V)$ defined by $\boxed{\mathcal{F}(X,Y) = [X_H, Y_H]_V}$ $(*)$

projection to H — proj. to V

for X, Y two vector fields on E.

Exercise: check that rhs of $(*)$ is $C^\infty(E)$-linear in X, Y, so that it does indeed define a 2-form

- **Parallel transport** (or "holonomy")

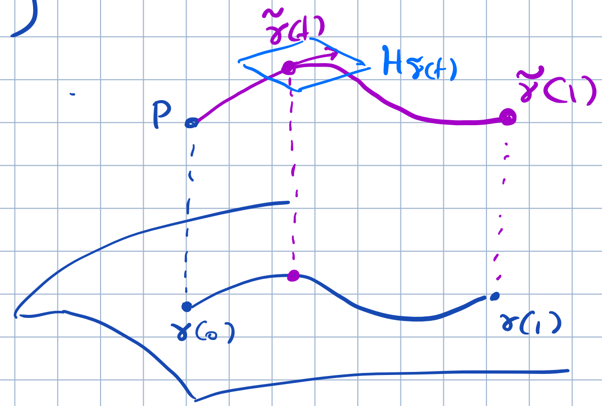
Let $\gamma : \underset{[0,1]}{I} \to M$ be a curve on the base, define the parallel transport as

$$\mathrm{Hol}_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$$
$$p \longmapsto \tilde{\gamma}(1) \qquad \text{where } \tilde{\gamma} : I \to E \text{ is the horizontal lift of } \gamma \text{ starting at } p.$$

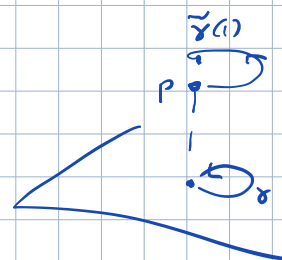
I.e. $\cdot\ \tilde{\gamma}(t) \in E_{\gamma(t)}$ (or: $\begin{tikzcd} & E \arrow[d, "\pi"] \\ I \arrow[ur, "\tilde{\gamma}"] \arrow[r] & M \end{tikzcd}$)

$\cdot\ \dot{\tilde{\gamma}}(t) \in H_{\tilde{\gamma}(t)}$ ODE

$\cdot\ \tilde{\gamma}(0) = p$ init. cond.



- Curvature measures how far is $\tilde{\gamma}(1)$ from p for γ a small closed loop.



In fact, for γ a small parallelogram with sides εu, εv, with $u, v \in T_xM$,

$$\mathrm{Hol}_\gamma : p \mapsto p + \underbrace{\varepsilon^2 F(\tilde{u}_p, \tilde{v}_p)}_{\in V_p \text{ - vertical shift of } p} + o(\varepsilon^2)$$

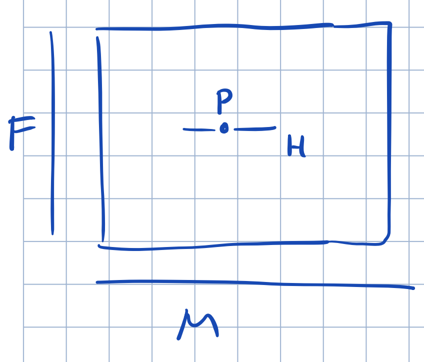
$\tilde{u}_p, \tilde{v}_p$: hor. lifts of u, v to H_p

- A connection has zero curvature (is "flat") iff H is integrable ($[H,H] \subset H$).

Then, by Frobenius theorem, H is tangent to a foliation of E.

Ex: A trivial bundle $\begin{tikzcd} M \times F \arrow[d] \arrow[dr] & \\ M & F \end{tikzcd}$ has the "trivial connection"

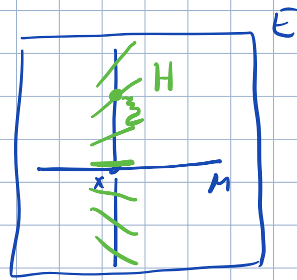
$H = \ker(d\, \mathrm{proj}_2)$



Connection in a vector bundle

For $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ a vector bundle, one considers Ehresmann connections inducing <u>linear</u> holonomy maps.

In other words, $H_{(x,\xi)}$ depends linearly on $\xi \in E_x$.



In a trivializing nbhd $(U_\alpha, \varphi_\alpha : U_\alpha \times \overset{\swarrow \mathbb{R}^k}{W} \to E|_U)$, $x \in U$, $v \in W$,

$$H_{(x,v)} = \{(\theta, -A_\alpha(\theta) \circ v) \mid \theta \in T_x M\} \subset T_{(x,v)}(M \times W)$$
$$= \operatorname{graph}(-A_\alpha \circ v)$$

where $A_\alpha \in \Omega^1(U_\alpha, \operatorname{End} W)$ – the (matrix-valued) 1-form of the connection in a local trivialization.

(and $\underset{T_pE}{\overset{\cap}{H_p}} = d_{(x,v)}\varphi \left(H_{(x,v)} \right)$ for $p = (x,v)$)

Covariant derivative

Covar. derivative of a section along a vector field:

$$(\nabla_X \sigma)(x) = \frac{d}{dt}\Big|_{t=0} \underbrace{\left(\mathrm{Hol}_{\gamma_0^t}\right)^{-1}}_{\substack{E_x \to E_{\gamma(t)} \\ p = \tilde{\gamma}(0) \mapsto \tilde{\gamma}(t)}} \sigma(\gamma(t))$$

X: v.f. on M, σ: section of E

for $\gamma : [0,1] \to M$ a curve starting at x with initial velocity $\dot{\gamma}(0) = X(x)$

$\rightsquigarrow \nabla : \Gamma(M,E) \to \Omega^1(M,E)$ – (1st order) differential operator satisfying Leibniz rule $\nabla(f\sigma) = f \cdot \nabla\sigma + df \cdot \sigma$

(with $\nabla_X \sigma = \iota_X(\nabla\sigma)$)

One can extend ∇ to a operator

$$\nabla : \Omega^p(M,E) \to \Omega^{p+1}(M,E), \quad 0 \le p < \dim M ,$$

by Leibniz rule $\nabla(\alpha\sigma) = d\alpha \cdot \sigma + (-1)^p \alpha \wedge \nabla\sigma$ (+ linearity)

p-form section