Covariant derivative

Covar. derivative of a section along a vector field:

$$\left(\nabla_X \sigma\right)(x) = \frac{d}{dt}\Big|_{t=0} \underbrace{\left(\mathrm{Hol}_{\gamma_0^t}\right)^{-1}}_{} \sigma(\gamma(t))$$

v.f. on M section of E

$E_x \to E_{\gamma(t)}$

$p = \tilde{\gamma}(0) \mapsto \tilde{\gamma}(t)$

for $\gamma : [0,1] \to M$ a curve starting at x with initial velocity $\dot{\gamma}(0) = X(x)$

$\rightsquigarrow$ $\nabla : \Gamma(M,E) \to \Omega^1(M,E)$ - (1st order) differential operator

(with $\nabla_X \sigma = \iota_X(\nabla\sigma)$)   satisfying Leibniz rule $\nabla(f\sigma) = f \cdot \nabla\sigma + df \cdot \sigma$

One can extend ∇ to a operator

$$\nabla : \Omega^p(M,E) \to \Omega^{p+1}(M,E), \quad 0 \le p < \dim M,$$

by Leibniz rule $\underset{\substack{\uparrow \\ p\text{-form}}}{\nabla(\alpha}\ \underset{\substack{\uparrow \\ \text{section}}}{\sigma)} = d\alpha \cdot \sigma + (-1)^p \alpha \wedge \nabla \sigma$ (+ linearity)

Locally, in a trivializing nbhd $(U_\alpha, \varphi_\alpha)$, one has

$$\nabla : \sigma_\alpha(x) \longmapsto (d + A_\alpha)\, \sigma_\alpha(x)$$

$$\begin{tikzcd} \pi^{-1}(U_\alpha) & U_\alpha \times W \arrow[l, "\varphi_\alpha"'] \\ U_\alpha \arrow[u, "\sigma"] \arrow[ur, "x \mapsto (x, \sigma_\alpha(x))"'] & \end{tikzcd}$$

local matrix-valued connection 1-form

$A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}\, W)$

← same A_α as in

$H_{(x,v)} = \mathrm{graph}(-A_\alpha \circ v)$

$T_x M \to T_v W$

(Ehresmann description)

or: $\sum_a e_a \sigma^a(x) \longmapsto \sum_a e_a \left(d\sigma^a(x) + \sum_b A^a{}_b(x)\, \sigma^b(x) \right)$

e_a: basis sections (over U_α)

- $\nabla^2 = F \wedge$

  with $F \in \Omega^2(M, \mathrm{End}\, E)$ - curvature 2-form

(relation to Ehresmann curvature 2-form:

$\mathcal{F} \in \Omega^2(E, V)$, $\quad \mathcal{F}_{(x,\xi)}(X,Y) = F_x(d\pi(X), d\pi(Y)) \circ \xi$)

Exercise: ① Show that locally one has $F = dA + \frac{1}{2}[A, A]$

② Show that on an overlap $U_\alpha \cap U_\beta$ one has

$A_\beta = t_{\alpha\beta}^{-1} A_\alpha t_{\alpha\beta} + t_{\alpha\beta}^{-1} d t_{\alpha\beta}$ with $t_{\alpha\beta}$ the transition function

Lemma a) Connections in $\begin{array}{c}E\\ \downarrow\\ M\end{array}$ exist

b) the space of connections in $\begin{array}{c}E\\ \downarrow\\ M\end{array}$ is an affine space modeled on $\Omega^1(M, End(E))$

Proof (a) Let $\{U_\alpha, \varphi_\alpha\}$ be a loc. triv. and $\{\psi_\alpha\}$ - partition of unity subordinate to $\{U_\alpha\}$.

Choose $\underbrace{\text{1-form}}_{\text{fiber-linear}}$ $\mathcal{A}_\alpha \in \Omega^1(\pi^{-1}(U_\alpha), V)$ with $\mathcal{A}_\alpha|_V = id_V$. E.g. $\mathcal{A}_\alpha = (\varphi_\alpha^{-1})^* \underbrace{\tilde{\mathcal{A}}_\alpha}$

$(\theta, x) \longmapsto x$

$\in T_{(x,v)}E \quad \begin{array}{c}\cap\\ T^{vert}_{(x,v)}E\end{array}$

Set $\mathcal{A} = \sum_\alpha \pi^*(\psi_\alpha)\, \mathcal{A}_\alpha$

(b) Given two connections ∇_1, ∇_2 (covar. derivatives) in E, $\nabla_2 - \nabla_1 = \alpha\wedge$ - multiplication operator by some 1-form $\alpha \in \Omega^1(M, End(E))$

($C^\infty(M)$-linearity follows from Leibniz for $\nabla_{1,2}$)

and $\nabla_2 = \nabla_1 + \alpha\wedge$ is a connection for ∇_1 a con. and $\alpha \in \Omega^1(M, End E)$ any 1-form.

$\square$

Connections in a principal bundle

- An Ehresmann connection on $\begin{array}{c}P \circlearrowleft G\\ \downarrow\\ M\end{array}$ which is G-equivariant,

i.e. $H_{p\cdot g} = (dR_g)_p H_p$

$R_g : P \to P$

$y \mapsto y\cdot g$

Connection 1-form $\mathcal{A} \in \Omega^1(P, \mathfrak{g})$ ($\mathfrak{g}$ = Lie(G))

satisfies ① Equivariance $\quad R_g^* \mathcal{A} = Ad_{g^{-1}}(\mathcal{A})$

② Normalization $\quad \iota_{X_\xi} \mathcal{A} = \xi$ with any $\xi \in \mathfrak{g}$ and X_ξ the corresponding fundamental vect. field on P

Curvature: $\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}] \in \Omega^2(P,\mathfrak{g})$ — G-equivariant, horizontal 2-form

It corresponds to a 2-form on the base $F \in \Omega^2(M, \underbrace{ad(P)}_{P \times_G^{ad} \mathfrak{g}})$ — "adjoint bundle"

- In a trivializing nbhd U_α with section s_α,

$F_\alpha := \sigma_\alpha^* \mathcal{F} \in \Omega^2(U_\alpha, \mathfrak{g})$

on $U_\alpha \cap U_\beta$, $\quad F_\beta = \underbrace{ad(t_{\beta\alpha})}_{\text{correct transition function for the bundle } ad(P)} \circ F_\alpha$ ← from G-equivariance of $\mathcal{F}$

- Another way to compare F and $\mathcal{F}$:

$F_x(d\pi(X), d\pi(Y)) = \underbrace{[(p, \mathcal{F}_p(X,Y))]}_{\in\, ad(P)_x}$ ← G-orbit $\quad$ for $p \in P_x$, $X, Y \in T_p P$

In fact, one has a general

Lemma

$$\Omega^k(P,V)^{\substack{\text{horizontal}\\ G\text{-equivariant}}} \simeq \Omega^k(M, P \times_G V)$$

V: vect. space carrying a rep. of G

Cor. Space of connections on $P \circlearrowleft G \to M$ is an affine space modeled on $\Omega^1(M, ad(P))$

Proof: A difference of two connection 1-forms $\mathcal{A}_2 - \mathcal{A}_1$ is equivar. and horizontal (from $\iota_{X_\xi}\mathcal{A}_{1,2} = \xi$)

$$\Rightarrow \mathcal{A}_2 - \mathcal{A}_1 \in \Omega^1(P,\mathfrak{g})^{\substack{\text{equiv}\\ \text{horiz}}} \simeq \Omega^1(M, ad(P))$$

$\square$

Remark: Reduction of structure group - examples.

- Given a real v.b. $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$, we can choose a metric in fibers $g_x: E_x \otimes E_x \to \mathbb{R}$ (Euclidean; symmetric, positive-definite) smoothly depending on $x \in M$. (I.e., $g \in \Gamma(M, \mathrm{Sym}^2 E^*)$)
  - Can choose loc. trivialization to respect g (i.e., $\varphi_\alpha^* g =$ stand. metric on fiber $\mathbb{R}^k$)

  Then transition functions $t_{\alpha\beta}(x) \in O(k) \subset GL(k)$
  - "reduction of structure group to $O(k)$."
- Likewise, for a $\mathbb{C}$ v.b. $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$, one can choose a fiberwise Hermitian metric.

  Then $t_{\alpha\beta}(x) \in U(k) \subset GL(k, \mathbb{C})$ - reduction of structure group to $U(k)$
- A rk $= k$ real v.b. E is called <u>orientable</u> if $\Lambda^k E$ is trivial.

  Then, one can choose a fiber volume form $\omega \in \Gamma(M, \Lambda^k E^*)$

  $\sim$ reduction of structure group to $SL(k) \subset GL(k)$
- Trivialization of E $\sim$ reduction of str. group to $\mathbb{1} \subset GL(k)$

Generally: if $H \subset G$ a subgroup, an "H-structure" on a G-bundle $\begin{matrix} P \circlearrowleft G \\ \downarrow \\ M \end{matrix}$ is an H-bundle $\begin{matrix} P' \circlearrowleft H \\ \downarrow \\ M \end{matrix}$ and H-equiv. inclusion

$$\begin{tikzcd} P' \circlearrowleft H \arrow[r, hook, "i"] \arrow[d] & P \circlearrowleft H \arrow[d] \\ M \arrow[r, "id"] & M \end{tikzcd}$$

bundle with str. grp reduced to H

Ex: Given a fiber metric on a v.b. $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$, one has

- reduction of str. group from $GL(k)$ to $O(k)$

bundle of orthonormal frames

$$\begin{tikzcd} F_{o/n} E \arrow[rr, hook] \arrow[dr] & & FE \arrow[dl] \\ & M & \end{tikzcd}$$