LAST TIME

Pontrjagin classes:

$$\begin{tikzcd}
 & \text{real v.b.} \arrow[dr, "\otimes \mathbb{C}"] & \\
\text{oriented v.b.} \arrow[ur] & & \text{cx. v.b.} \arrow[ll, "(\ )_{\mathbb{R}}"]
\end{tikzcd}$$

p_i (real v.b.), c_i (cx. v.b.), e (oriented v.b.)

$$p_i(E) := (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(M;\mathbb{Z})$$

(E: real v.b.)

- $(E \otimes \mathbb{C})_{\mathbb{R}} = E \oplus E$ (E: real v.b.)
- $\omega_{\mathbb{R}} \otimes \mathbb{C} \simeq \omega \oplus \bar{\omega}$

$$\underbrace{1 - p_1 + p_2 - \dots \pm p_k}_{p_i(\omega_{\mathbb{R}})} = \underbrace{(1 - c_1 + c_2 - \dots \pm c_k)}_{c(\bar{\omega})}\underbrace{(1 + c_1 + c_2 + \dots + c_k)}_{c(\omega)}$$

Ex: $p_i(\underset{\substack{\text{as a real v.b.,}\\ \text{rk}=2n}}{T}\mathbb{CP}^n) = \binom{n+1}{2i} a^{2i}$, $\quad i = 1, \dots, \left[\frac{n}{2}\right]$

$a = -c_1\left(\begin{smallmatrix}\tau \\ \downarrow \\ \mathbb{CP}^n\end{smallmatrix}\right) \in H^2(\mathbb{CP}^n; \mathbb{Z}_2)$

n	$p(\mathbb{CP}^n)$
1	1
2	$1 + \boxed{3a^2}$
3	$1 + \boxed{4a^2}$
4	$1 + \boxed{5a^2} + \boxed{10a^4}$

p_1 $\quad$ p_2

Lemma If $E \downarrow M$ is an oriented v.bun ($rk = 2k$), then $p_k(E) = e(E) \cup e(E)$

↑ Euler class

$\ulcorner$ $p_k(E) = \pm c_{2k}(E \otimes \mathbb{C}) = \pm e(\underbrace{(E\otimes\mathbb{C})_{\mathbb{R}}}_{E \oplus E}) = \pm e(E) \cup e(E)$ $\lrcorner$

Chern numbers Let X - cpt complex mfd, $\dim_{\mathbb{C}} X = n$.

Then for each partition $n = i_1 + \dots + i_r$,

we have a Chern number $c_{i_1} \cdots c_{i_r}[X] := \langle c_{i_1}(TX) \cdots c_{i_r}(TX), \underset{\substack{\cap \\ H_{2n}(X;\mathbb{Z})}}{\overset{\text{fund. class}}{[X]}} \rangle \in \mathbb{Z}$

$= \langle c_{i_1}(\tau^n) \cdots c_{i_r}(\tau^n), f_*[X] \rangle$

↑ classifying map $f: X \to Gr_{\mathbb{C}}(n, \infty)$

Ex: for $X = \mathbb{CP}^n$, $c_i = \binom{n+1}{i} a^i$, $\langle a^{2n}, [\mathbb{CP}^n] \rangle = 1$

$\Rightarrow$ $\boxed{c_{i_1} \cdots c_{i_r}[\mathbb{CP}^n] = \binom{n+1}{i_1} \cdots \binom{n+1}{i_r}}$ for $i_1 + \dots + i_r = n$.

Pontrjagin numbers M smooth, cpt, oriented mfd, $\dim M = 4n$

for a partition $n = i_1 + \dots + i_p$ one has a Pontrjagin number

$p_{i_1} \cdots p_{i_r}[M] := \langle p_{i_1}(TM) \cdots p_{i_r}(TM), [M] \rangle$

Ex: $p_{i_1} \cdots p_{i_r}[\mathbb{CP}^{2n}] = \binom{2n+1}{i_1} \cdots \binom{2n+1}{i_r}$, $i_1 + \dots + i_r = n$

- under a change of orientation $M \to M^{op}$, p_i's do not change, but $[M]$ changes sign, thus all Pontrjagin #'s change sign

$\Rightarrow$ if some Pontrjagin number $p_{i_1} \cdots p_{i_r}[M]$ is nonzero, then M cannot possess any orientation reversing diffeomorphism!

Ex: $\mathbb{CP}^{2n}$ doesn't admit or.-rev. diffeo

while $\mathbb{CP}^{2n+1}$ does admit (coming from cx conjugation)

Thm (Pontrjagin) If an oriented cpt $4n$-mfd M is the boundary of an oriented $(4n+1)$-mfd N, then all Pontrjagin numbers of M are zero. [proof - as in non-oriented, Stiefel-Whitney case]

- two cpt, or n-mfds M_1, M_2 belong to the same oriented cobordism class iff $M_1 \amalg (-M_2) = \partial N$ for some cpt or. $(n+1)$-mfd N
  ($-$: reversed or.)

$\Omega_n = \{\text{cpt or. } n\text{-mfds}\}/\text{or. cobordism}$

Ω_* is a ring with $+ = \amalg$, $\cdot = \times$ — or. cobordism ring.

- $p_{i_1} \cdots p_{i_r} : \Omega_{4n} \longrightarrow \mathbb{Z}$ is a group homom.
  $i_1 + \dots + i_r = n$
- products $\mathbb{C}P^{2i_1} \times \dots \mathbb{C}P^{2i_r}$, $i_1 + \dots + i_r = n$ are lin. indep. in Ω_{4n} (distinguished by Pontrjagin numbers)
  (but might not generate all Ω_{4n})

$\Omega_0 = \mathbb{Z}$ ← generated by $[pt^+]$

$\Omega_1 = 0$

$\Omega_2 = 0$

$\Omega_3 = 0$

$\Omega_4 = \mathbb{Z}$ ← generated by $[\mathbb{C}P^2]$

$\Omega_5 = \mathbb{Z}_2$

$\Omega_6 = 0$

$\Omega_7 = 0$

$\Omega_8 = \mathbb{Z} \oplus \mathbb{Z}$ ← gen. by $[\mathbb{C}P^4], [\mathbb{C}P^2 \times \mathbb{C}P^2]$

$\Omega_9 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Omega_{10} = \mathbb{Z}_2$

$\Omega_{11} = \mathbb{Z}_2$

<u>Thm</u> $\Omega_* \otimes \mathbb{Q} \simeq \mathbb{Q}[x_1, x_2, x_3, \dots]$ polynomial algebra

$$x_i = [\mathbb{C}P^{2i}]$$

<u>Oriented Grassmannian</u>

$\widetilde{Gr}_{\mathbb{R}}(k,\infty)$ ($= BSO(k)$) – Grassmannian of <u>oriented</u> k-planes in $\mathbb{R}^\infty$. One has

$$\begin{array}{ll} \widetilde{Gr}(k,\infty) & = BSO(k) \\ \downarrow 2:1 & \\ Gr(k,\infty) & = BO(k) \end{array}$$

$\widetilde{Gr}$ comes with its taut. bundle $\begin{array}{c}\tilde{\tau}^k \\ \downarrow \\ \widetilde{Gr}(k,\infty)\end{array}$

for $\begin{array}{c}E \\ \downarrow \\ M\end{array}$ an or. v. bun. ($rk = k$), one has a classifying map $f: M \to \widetilde{Gr}(k,\infty)$ s.t. $E \sim f^* \tilde{\tau}^k$
($\sim$: or. preserving bun. iso)

Thm (12.4 MS): $H^\bullet(\widetilde{Gr}(k,\infty);\mathbb{Z}_2) = \mathbb{Z}_2[\tilde{\omega}_2,\dots,\tilde{\omega}_k]$, $\tilde{\omega}_i = \omega_i(\tilde{\tau}^k) = p^* \omega_i(\tau^k)$

in particular, $H^1(\widetilde{Gr}(k,\infty);\mathbb{Z}_2) = 0 \Rightarrow \omega_1(\tilde{\tau}^k) = 0 \Rightarrow$ $\boxed{\omega_1(E) = 0 \text{ for any orientable } E.}$

or any integrable domain with invertible 2 (com. ring w/o zero divisors)

Thm (15.9) Let $\Lambda = \mathbb{Z}[\frac{1}{2}]$ - coeff. ring with invertible 2.

$$H^\bullet(\widetilde{Gr}(k,\infty);\Lambda) = \Lambda\left[p_1(\tilde{\tau}^k),\dots,p_{[\frac{k}{2}]}(\tilde{\tau}^k), e(\tilde{\tau}^k)\right] \Big/ \begin{array}{ll} e=0 & \text{for } k \text{ odd} \\ e^2 = p_{\frac{k}{2}} & \text{for } k \text{ even} \end{array}$$

Classifying bundle & char. classes for principal G-bundles

ref: Ralph Cohen "Bundles, homotopy, and manifolds", sec. 4.2

def a top. space X is "aspherical" if $\pi_n(X) = 0$, $n \geq 0$
(homotopy groups)

Thm (Whitehead) if X has the homotopy type of a CW complex, then
X is aspherical $\Leftrightarrow$ X is contractible

fix a Lie group G

Thm (*) Let $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ be a G-bundle where E is aspherical. Then this bundle is universal,

i.e. for any top space M of the homotopy type of a CW complex, the map

$$\psi: [M,B] \longrightarrow \{G\text{-bun. over } M\}/_{\text{iso}} \quad \text{is a bijection.}$$
$$(f: M \to B) \longmapsto \begin{array}{c} f^*E \\ \downarrow \\ B \end{array}$$

$\psi^{-1}\left(\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M \end{array}\right) =:$ classifying map $f_P : M \to B$

Cor: If $\begin{array}{c} E_1 \circlearrowleft G \\ \downarrow \\ B_1 \end{array}$, $\begin{array}{c} E_2 \circlearrowleft G \\ \downarrow \\ B_2 \end{array}$ two universal G-bundles, then there exists a bundle map

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{h}"] \arrow[d] & E_2 \arrow[d] \\ B_1 \arrow[r, "h"'] & B_2 \end{tikzcd}$$

where h is a homotopy equivalence

Proof: $\begin{array}{c} E_2 \\ \downarrow \\ B_2 \end{array}$ universal $\Rightarrow \exists$ classifying map $h: B_1 \to B_2$ s.t. $E_1 \simeq h^* E_2$. Similarly, $\exists\, g: B_2 \to B_1$ s.t. $E_2 \simeq g^* E_1 \Rightarrow$ we have $B_1 \xrightarrow{h} B_2 \xrightarrow{g} B_1$ and $E_1 \simeq h^* g^* E_1 \underset{\text{Thm}^*}{\Rightarrow} gh \underset{\text{homotopic}}{\sim} id$. Likewise, $hg \sim id$ □

<u>Notation</u>: $\begin{matrix} EG \\ \downarrow \\ BG \end{matrix}$ - universal G-bundle. By Cor., BG is well-defined up to homotopy.

- One has $\pi_{n-1}(G) \simeq \pi_n(BG)$ iso of homotopy groups

(from LES of π_*(fibration):

$$\cdots \to \underset{\underset{G}{\parallel}}{\pi_n(F)} \to \underbrace{\pi_n(E)}_{\underset{0}{\parallel}} \to \pi_n(B) \to \underset{\underset{G}{\parallel}}{\pi_{n-1}(F)} \to \underbrace{\pi_{n-1}(E)}_{\underset{0}{\parallel}} \to \cdots$$
)

<u>Ex</u>: For G a <u>discrete</u> group, $BG = K(G,1)$ - Eilenberg-MacLane space

(recall: $X = K(G,n)$ if $\pi_i(X) = \begin{cases} G, & i = n \\ 0, & i \neq n \end{cases}$)

univ. bundle: $\begin{matrix} EG = \widetilde{K(G,1)} \circlearrowleft G \\ \downarrow \\ BG = K(G,1) \end{matrix}$ univ. cover

Ex: $G = \mathbb{Z}$ $\rightsquigarrow$ $BG = S^1$, $\begin{matrix} EG \\ \downarrow \\ BG \end{matrix} = \begin{matrix} \mathbb{R} \circlearrowleft \underset{x \mapsto x+k}{\mathbb{Z}} \\ \downarrow \\ S^1 = \mathbb{R}/\mathbb{Z} \end{matrix}$

Ex: $G = \mathbb{Z}_2 \longrightarrow$ $\begin{matrix} EG = S^\infty \circlearrowleft \mathbb{Z}_2 \text{ antipodal map} = \varinjlim_n S^n \quad (\text{under } S^1 \subset S^2 \subset S^3 \subset \cdots) \\ \downarrow \\ BG = \mathbb{RP}^\infty = S^\infty/\mathbb{Z}_2 \end{matrix}$

contractible!