characteristic classes of G-bundles:

a char. class of G-bundles is a map $\varphi : \left\{ \text{G-bundle } \begin{matrix} P \circlearrowleft G \\ \downarrow \\ M \end{matrix} \right\} \longrightarrow H^\bullet(M; \Lambda)$ ← fixed ring

with naturality property:

$$\begin{tikzcd} P' \circlearrowleft G \arrow[r, "\bar{f}"] \arrow[d] & P \circlearrowleft G \arrow[d] \\ M' \arrow[r, "f"] & M \end{tikzcd} \Rightarrow \varphi(P') = f^* \varphi(P)$$

we have

$$\left\{ \begin{matrix} \text{char. classes} \\ \text{of G-bundles} \end{matrix} \right\} \longleftrightarrow H^\bullet(BG; \Lambda) \quad \text{ring isomorphism}$$

$$\left(\begin{matrix} P \\ \downarrow \\ M \end{matrix} \longmapsto f_P^* \alpha \right) \longleftarrow\!\!\shortmid \ \alpha \in H^\bullet(BG)$$

f_P: classifying map; α: cohom class of BG

$$\text{char. class } \varphi \longmapsto \varphi \left(\begin{matrix} EG \\ \downarrow \\ BG \end{matrix} \right) \in H^\bullet(BG)$$

Rem For $G = GL(k, \mathbb{R})$, one has $\left\{ \begin{matrix} \text{rk } k \\ \text{vector bundles} \\ \text{over } M \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{G-bundles} \\ \text{over } M \end{matrix} \right\}$

$$\begin{matrix} E \\ \downarrow \\ M \end{matrix} \longmapsto \begin{matrix} F_E \\ \downarrow \\ M \end{matrix} \quad \text{frame bundle}$$

$$\begin{matrix} P \times_G \mathbb{R}^k \\ \downarrow \\ M \end{matrix} \longleftarrow\!\!\shortmid \begin{matrix} P \circlearrowleft G \\ \downarrow \\ M \end{matrix}$$

have the same classifying $f: M \to BG$

s.t. $E \simeq f^* \tau^k$, $P \simeq f^* \underbrace{EG}_{= V(k,\infty) = F(\tau^k)}$ ($F(\tau^k)$: frame bundle)

So: {char. classes of rk k v. bun} = {char. classes of $GL(k, \mathbb{R})$-bundles}

Recall: a connection in $\begin{array}{c} P \circlearrowleft G \\ \downarrow \pi \\ M \end{array}$ is $\mathcal{A} \in \Omega^1(P; \mathfrak{g})$ with

a) $\iota_{X_\alpha} \mathcal{A} = \alpha$ normalization

X_α: vert. v.field on P corresp. to $\alpha \in \mathfrak{g}$

b) G-equivariant

covariant derivative

$$\nabla = d + \mathrm{ad}(\mathcal{A})\wedge : \quad \Omega^\bullet(P; \mathfrak{g}) \to \Omega^{\bullet+1}(P; \mathfrak{g})$$

$$\begin{tikzcd}
\Omega^\bullet(P; \mathfrak{g})^{\text{basic}} \arrow[r] \arrow[d, phantom, "\cong"] & \Omega^{\bullet+1}(P; \mathfrak{g})^{\text{basic}} \arrow[d, phantom, "\cong"] \\
\Omega^\bullet(M, \mathrm{ad}(P)) \arrow[r] \arrow[u, bend left, "\pi^*"] & \Omega^{\bullet+1}(M, \mathrm{ad}(P))
\end{tikzcd}$$

basic: horizontal, G-equivar.

$$\nabla^2 = \mathrm{ad}(F)\wedge, \quad F \in \Omega^2(P; \mathfrak{g})^{\text{basic}}.$$

$$F = d\mathcal{A} + \tfrac{1}{2}[\mathcal{A}, \mathcal{A}]$$

Chern-Weil theory

Thm Given a (C^∞) principal bundle $\begin{matrix} P \circlearrowleft G \\ \downarrow \pi \\ M \end{matrix}$, there is a homomorphism of graded com. algebras

$$\Psi : \underbrace{(S^\bullet \mathfrak{g}^*)^G}_{} \longrightarrow H^\bullet(M;\Bbbk)$$

polynomials $p: \mathfrak{g} \to \Bbbk$, $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, for G a real or $\mathbb{C}$x Lie group
default, e.g. $GL(n,\mathbb{C})$
s.t. $p(\mathrm{Ad}_g \alpha) = p(\alpha)$ $\forall g \in G$, $\alpha \in \mathfrak{g}$

mapping

$$\underset{\text{poly. on } \mathfrak{g}}{\underset{\text{invar.}}{p}} \longmapsto \underbrace{[p(F_A)]}_{(\#)}$$

curvature 2-form $F_A \in \Omega^2(M, \mathrm{ad}(P))$ of an arbitrary connection ∇ on P.

· how to understand (#)?

a) Locally, in a loc. trivialization,

$$F_\alpha \in \Omega^2(U_\alpha, \mathfrak{g}) = \Omega^2(U_\alpha) \otimes \mathfrak{g}$$

So, $p(F_\alpha)$ is defined via change of scalars $\Bbbk \to \Omega^2(U_\alpha)$

if $p(\xi) = \sum_{a_1 \ldots a_r} p_{a_1 \ldots a_r} \xi^{a_1} \cdots \xi^{a_r}$ ($\xi = \sum \xi^a T_a$, basis vectors of $\mathfrak{g}$)
then $p(F_\alpha) := \sum p_{a_1 \ldots a_r} F_\alpha^{a_1} \wedge \cdots \wedge F_\alpha^{a_r}$ ($\sum F_\alpha^a T_a$)

in an overlap $U_\alpha \cap U_\beta$, $F_\beta = \mathrm{Ad}_{t_{\beta\alpha}} F_\alpha$

and $p(F_\beta) = p(F_\alpha)$ due to G-invariance of p

b) $\mathcal{F}_A \in \underbrace{\Omega^2(P;\mathfrak{g})}_{\Omega^2(P)\otimes\mathfrak{g}}^{G\text{-equiv horiz}}$ – curvature as a form on total space

· $\underset{\text{poly of degree } r}{f}(\mathcal{F}_A) \in \Omega^{2r}(P)^{G\text{-inv. horiz}}$ $\Rightarrow$ $f(\mathcal{F}_A) = \pi^* \underbrace{\omega}_{\text{some } 2r\text{-form on } M}$

$$\omega := f(F_A) \in \Omega^{2r}(M)$$

Proof (i) why $p(F_A)$ is a closed form?

on the total space: Bianchi identity: $(d + \mathrm{ad}_A)\underbrace{\mathcal{F}_A}_{dA + \frac{1}{2}[A,A]} = 0$ – on total space

$[\nabla,[\nabla,\nabla]] = [[\nabla,\nabla],\nabla] - [\nabla,[\nabla,\nabla]]$ ($[\nabla,\nabla] = \mathcal{F}^\wedge$, $\nabla\mathcal{F}^\wedge$) $= -2[\nabla,[\nabla,\nabla]]$ $\Rightarrow [\nabla,[\nabla,\nabla]] = 0$

$\underbrace{\nabla}_{} F_A = 0$ on the base

$\Omega^\bullet(M, \mathrm{ad}\,P) \to \Omega^{\bullet+1}(M, \mathrm{ad}\,P)$
cover. derivative

Say, $\underset{\underset{\mathfrak{g}}{\cap}}{p(\alpha)} = \langle \underset{\underset{(\mathrm{Sym}^r \mathfrak{g}^*)^G}{\cap}}{p}, \underbrace{\alpha \otimes \cdots \otimes \alpha}_{r} \rangle = \sum_{a_1 \ldots a_r} p_{a_1 \ldots a_r} \alpha^{a_1} \cdots \alpha^{a_r}$ for $\alpha = \alpha^a \underset{\text{basis in } \mathfrak{g}}{T_a}$

then $\overset{\text{on total space}}{d\, p(\mathcal{F}_A)} = \langle p, \sum_{i=1}^{r} \mathcal{F} \otimes \dots \underset{\substack{\uparrow \\ i^{th} \text{ place}}}{d\mathcal{F}} \otimes \dots \mathcal{F} \rangle =$

$$= \langle p, \sum_{i=1}^{r} \mathcal{F} \otimes \dots \otimes \underbrace{(d + ad_A)\mathcal{F}}_{0 \text{ by Bianchi}} \otimes \dots \otimes \mathcal{F} \rangle - \underbrace{\langle p, \underbrace{\sum_{i=1}^{r} \mathcal{F} \otimes \dots ad_A \mathcal{F} \dots \otimes \mathcal{F}}_{ad_A(\mathcal{F} \otimes \dots \otimes \mathcal{F})} \rangle}_{0 \text{ by } G\text{-invariance of } p.}$$

$= 0$

So: $d\,p(\mathcal{F}_A) = 0 \Rightarrow d\,p(F_A) = 0$ on the base

(ii) If A' is a connection different from A, then $A' = A + \beta$, $\beta \in \Omega^1(P; \mathfrak{g})^{G\text{-equiv horiz}}$

Say $A_t = (1-t)A + tA'$, $t \in [0,1]$

then $\boxed{\frac{d}{dt}\mathcal{F}_t = (d + ad_{A_t})\underbrace{\dot{A}_t}_{\beta}}$

(a special case of
$\mathcal{F}_{A+\beta} = \mathcal{F}_A + (d + ad_A)\beta + \frac{1}{2}[\beta, \beta]$)

Then: $\frac{d}{dt} p(\mathcal{F}_t) = \langle p, \sum_{i=1}^{r} \mathcal{F}_t \otimes \dots \underbrace{\frac{d}{dt}\mathcal{F}_t}_{(d + ad_{A_t})\dot{A}_t} \dots \otimes \mathcal{F}_t \rangle =$

$$= \langle p, (d + \underset{\substack{\uparrow \\ \text{can ignore by } G\text{-invar. of } p}}{ad_{A_t}})(\sum_{i=1}^{r} \mathcal{F}_t \otimes \dots \dot{A}_t \dots \otimes \mathcal{F}_t) \rangle = d \langle p, \sum_{i=1}^{r} \mathcal{F}_t \otimes \dots \dot{A}_t \dots \otimes \mathcal{F}_t \rangle$$

- d-exact form.

So: $p(\mathcal{F}_{A'}) - p(\mathcal{F}_A) = d(\dots)$ $\Rightarrow$ de Rham cohomology class $[p(\mathcal{F}_A)]$ does not depend on the connection A !

(iii) for p_1, p_2 two G-invar. polynomials on $\mathfrak{g}$,

$$[(p_1 \cdot p_2)(\mathcal{F}_A)] = [p_1(\mathcal{F}_A) \wedge p_2(\mathcal{F}_A)] = [p_1(\mathcal{F}_A)] \cup [p_2(\mathcal{F}_A)]$$

So, Ψ is compatible with products.

Similarly, for sums $\Rightarrow$ Ψ is a homomorphism. $\square$

Naturality: for
$$\begin{tikzcd} P'^{\curvearrowleft G} \arrow[r, "\tilde f"] \arrow[d] & P^{\curvearrowleft G} \arrow[d] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$
a morphism of G-bundles, if A is a conn. on P then $\tilde f^* A$ is a connection on P'!

$\text{Curvature}(\tilde f^* A) = \tilde f^* \text{curvature}(A)$

Thus, $\underset{\substack{\uparrow \\ \text{Chern-Weil map}}}{\Psi}(\underset{\substack{\text{invar. polynomial} \\ \text{on } \mathfrak{g}}}{p})\left(\begin{smallmatrix} P' \\ \downarrow \\ M' \end{smallmatrix}\right) = p(F_{\tilde f^* A}) = p(\tilde f^* F_A) = f^* p(F_A) =$

$\Omega^\bullet_{cl}(P')^{basic} \cong \Omega^\bullet_{cl}(M')$

$= f^* \Psi(p)\left(\begin{smallmatrix} P \\ \downarrow \\ M \end{smallmatrix}\right)$

So, $\Psi(p)$ is a char. class!

- Evaluating $\Psi(p)$ on $\begin{smallmatrix} EG \\ \downarrow \\ BG \end{smallmatrix}$ (assuming this is realized as a (direct limit of) smooth G-bundles), we get a map $\hat\Psi : (S^\bullet \mathfrak{g}^*)^G \longrightarrow H^{2\bullet}(BG; k)$ — a ring homomorphism (*)

Theorem For G a compact group, $\hat\Psi$ is an isomorphism $(S^\bullet \mathfrak{g}^*)^G \xrightarrow{\sim} H^{2\bullet}(BG; k)$

Example: $G = GL(n, \mathbb{C})$

$X \in \mathfrak{g}$ can be conjugated to $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

where $D \sim D'$ (conjugate) iff $\{\lambda_i\}$ is a permutation of $\{\lambda_i'\}$ ($\subset \mathbb{C}$)

$\text{Inv}(\mathfrak{g}) := (S^\bullet \mathfrak{g}^*)^G = \{$symmetric polynomials on eigenvalues $\lambda_1 \dots \lambda_n\}$

$= \mathbb{C}[\sigma_1, \dots, \sigma_n]$

elem. sym. polynomials of eigenvalues $\sigma_r(\lambda_1, \dots, \lambda_n) = \sum_{1 \le i_1 < \dots < i_r \le n} \lambda_{i_1} \cdots \lambda_{i_r}$

Ex: $\operatorname{tr} X^k$, $\det X$, $\det(I + \underset{\substack{\uparrow \\ \text{parameter}}}{\mu} X) \in \text{Inv}(\mathfrak{g})$ ($X \in \mathfrak{g}$)

$\operatorname{tr} X = \lambda_1 + \dots + \lambda_n = \sigma_1$

$\operatorname{tr} X^2 = \underbrace{\lambda_1^2 + \dots + \lambda_n^2}_{s_2(\lambda_1 \dots \lambda_n)} = \sigma_1^2 - 2\sigma_2$

$\det X = \sigma_n$, $\det(I + \mu X) = 1 + \mu \sigma_1 + \mu^2 \sigma_2 + \dots + \mu^n \sigma_n$.

$\quad \overset{\shortparallel}{\prod_i (1 + \mu \lambda_i)}$

②

- We can replace $G = \underset{\Bbbk=\mathbb{C}}{GL(n,\mathbb{C})}$ with $\underset{\Bbbk=\mathbb{R}}{U(n)}$ then eigenvalues $\lambda_i \in i\mathbb{R}$ of $X \in \mathfrak{u}(n)$ purely imaginary $X^\dagger = -X$

(For $U(n)$, $\Bbbk=\mathbb{R}$ but $\operatorname{tr} X^k$ is not a real polynomial; $i^k \operatorname{tr} X^k$ is real!)

- $\underset{\substack{\uparrow \\ X \mapsto \operatorname{tr} X}}{\Psi(c_1)} \left(\begin{array}{c} P^{2G} \\ \downarrow \\ M \end{array} \right) = \operatorname{tr} F_A \in \Omega^2_{c\ell}(M)$

some complex coeff, independent of P

$$= f_P^* \underbrace{\hat{\Psi}(c_1)}_{\in H^2(BG,\mathbb{C}) = \mathbb{C}\otimes H^2(BG,\mathbb{Z})} = f_P^*(\varkappa\, c_1^{\mathbb{C}}) = \varkappa\, c_1^{\mathbb{C}}(P)$$

$= \mathbb{C}\cdot c_1$ ← 1st Chern class

image of $c_1 \in H^2(BG,\mathbb{Z})$ under $H^2(BG,\mathbb{Z}) \to H^2(BG,\mathbb{C})$

- By Exercise II/5a, for $\begin{array}{c}\tau\\ \downarrow \\ \mathbb{C}P^1\end{array}$, $\int_{\mathbb{C}P^1} \underbrace{\operatorname{tr} F_A}_{\Psi(c_1)} = 2\pi i = \varkappa \cdot \underbrace{\langle c_1(\tau), [\mathbb{C}P^1] \rangle}_{-1}$

$\Rightarrow \varkappa = -2\pi i$

(cf. $(\tau^{\otimes(-2)})_{\mathbb{R}} \cong T\mathbb{C}P^1$ and $\chi(\mathbb{C}P^1) = c_1[\mathbb{C}P^1] = +2$)

$\Rightarrow$ $\left[-\frac{1}{2\pi i} \operatorname{tr} F_A \right] \in H^2_{\text{deRham}}(M)$ represents the first Chern class, $c_1^{\mathbb{C}}(P)$

In particular, this class is in the image of $H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{C})$

Thus, $\int_{\Sigma \subset M} -\frac{1}{2\pi i} \operatorname{tr} F_A = \langle c_1(P), [\Sigma] \rangle \in \mathbb{Z}$

Σ: immersed closed surface

i.e. all periods (on closed 2d surfaces in M) are integers!